MATHEMATICS TEACHING

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MATHEMATICS TEACHING

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EDITORIAL

PROGRESS REPORT

The first issue of *Mathematics Teaching* appeared in November 1955—a small booklet of twenty pages issued by the nucleus of a small Association, and published (let it be admitted) with fears as to whether we could really afford it and misgivings as to whether we could continue to maintain its production twice a year. In the Editorial of that issue we expressed the view that there was a need for a magazine such as ours. It is interesting to browse through the issues since then and note how our faith in the magazine was justified, and how each issue has shown an improvement in either quantity or quality of articles and format; one can see quite clearly the process of continued development.

For example, in the present issue, we use colour illustrations for the first time, and our last issue was the first one to have 72 pages. In November 1957 we commenced publication three times a year, while with our next issue (November 1961) we commence publication four times a year. In future *Mathematics Teaching* will appear in February, May, August and November.

Nor do we intend that this is the end of the story. Plans are already in hand for many new features which will continue our development in the future and bring to teachers of mathematics at all levels the unique service which they have come to expect from us.

So much for the credit side; what of the debit? For improvements can not be achieved without costs. It was not just the first issue which caused fears about our financial situation; throughout the history of the magazine there has been the restricting influence of a tight budget. On more than one occasion developments have been slowed down because of lack of finance; in fact, when one examines the accounts it is rather surprising that development has been so rapid and continuous. But now we come again to a period when further progress cannot be achieved without the guarantee of greater income. It is because of this that the Association has decided to raise its subscription from 10s. per year to £1, commencing with the subscription for 1962. A subscription of ten shillings is being maintained for student members in Training Colleges, etc.

The past subscription fee was extremely low, and the new fee of £1 (especially when one remembers the Income Tax allowance obtainable) is not high for membership of a professional body. With the increased income the Association is planning to make various improvements in such matters as reduced Conference fees to members, financial grants to branches, initiation of a research programme, etc., as well as larger issues of *Mathematics Teaching* four times a year.

We hope that members of the Association will continue their membership at the new rate, and will also encourage other people to join the Association. Increasing the membership of the Association is just as important as increasing the membership fee, since both are important sources of revenue, and the more money we have to

spend the better are the services which we can give to new and existing members. This is work which every member can do; may we appeal to *you* to help in recruitment? After all, we are certain that those whom you recruit will benefit from the introduction and will come to thank you for it in the future.

In six years we have established ourselves as one of the leading mathematical journals. We look to our readers to help us to extend our influence and serve the cause of better teaching of mathematics.

MR. CYRIL HOPE

President of the Association, 1961.

Most people who have attended courses run by the Association will have come under the influence, at some time or other, of a stimulating lecturer who combines challenge of outmoded ideas, and inspiration to new thinking with a jovial, good-humoured 'shirt-sleeved' delivery; this is Mr. Cyril Hope. Those of our readers who have not had the good fortune to meet Mr. Hope in this way will know something of him from his writings in this journal. From time to time we publish articles which he has written and which teem with ideas such as we find in his lectures, while his book reviews are a valuable feature of our columns; he is, in fact, our Book Review Editor.

Educated at Leigh Grammar School in Lancashire, he took a B.Sc. degree in Mathematics at Manchester University and taught in various Grammar Schools as well as doing industrial work with I.C.I. before coming to Training College work. He has been at the City of Worcester Training College for the past twelve years, and in that time has built up there a sound and vigorous Mathematics Department whose work is well-known throughout the country. His work and reputation impose many demands on him (a factor which means that our Association often sees less of him than it would like), and his activities in recent years have included the Chairmanship of the A.T.C.D.E. Mathematics Section, his appointment as British delegate at the O.E.E.C. Conference on the Teaching of Mathematics, and as an 'International Expert' at the O.E.E.C. working party on a new curriculum for mathematics (held in Yugoslavia last year). In addition he has served on the Mathematical Association's Teacher Training Committee and their Modern Mathematics Committee.

Mathematics Teaching congratulates Mr. Hope on the richly-deserved honour of being elected President of the Association for the coming year. We must also congratulate the Association in choosing for its President a man who has done, and is doing, so much for the improvement of the teaching of our subject. We wish Mr. Hope a happy and successful term of office, and the Association a further period of development under his Presidency.

A PROBLEM IN VISUAL PRESENTATION

T. J. FLETCHER AND C. BIRTWISTLE

The opportunity of giving a demonstration lecture on closed circuit television at the National Education and Careers Exhibition at Olympia in May 1959, was a challenge to think about geometry in purely visual terms, and it crystallised out a problem which had been half-formulated in my mind for some time.

It is well-known that if a circle rolls on another the locus of any point on the circumference of the rolling circle is an epicycloid or hypocycloid according to whether it rolls outside or inside the fixed circle. Furthermore, referring to Figure I, a simple proof to be found in any elementary text on dynamics shows that the velocity of the moving point P is directed along the line PT. One most usually meets the proof for the case when the circle is rolling along a straight line, but the proof is equally valid when the circle rolls on any curve whatever. So if the circle rolls on the shaded curve whose shape is arbitrary, P has a locus L (say), PT is the tangent to this locus and PN the normal. (N is, of course, the instantaneous centre of the motion).

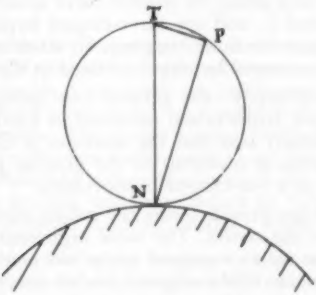

Fig. I

In particular, if the circle rolls inside another circle of twice the radius, T is always at the centre of the fixed circle and the locus of P is a diameter of the fixed circle. We may regard this locus as a two-cusped hypocycloid. (*See Fig. II overleaf*)

Now superimpose Figure I and Figure II, rolling the two circles of the second figure along the shaded curve of the first. PT is always tangent to the locus L, but PT coincides with a particular diameter of the larger circle. In other words, the tangents to L are the positions of a diameter of the larger circle; or, putting it yet another way, L may be generated as the envelope of a two-cusped hypocycloid carried along by a circle which is twice the radius of the original one.

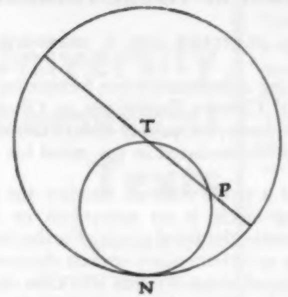

Fig. II

This may now be extended further. Let us take the radius of the circle on which P is mounted as unity, and call this 'Circle **1**'. A circle of n times the radius we will call 'Circle **n**'. If we roll Circle **1** and Circle **2** in step inside Circle **3**, P generates a three-cusped hypocycloid to which the diameter of Circle **2** is always a tangent. So now roll **1, 2,** and **3** in step along the shaded curve as we generate L. P generates L, the diameter of **2** touches L, and the three-cusped hypocycloid in Circle **3** also touches L (because they have the same tangent). In other words L is also generated as an envelope by the three-cusped hypocycloid fixed in Circle **3**.

We may continue indefinitely—the general case being that L is generated as an envelope by the n-cusped hypocycloid mounted in Circle-**n** as it rolls along the shaded curve. We have already said that the diameter of Circle **2** is to be regarded as a two-cusped hypocycloid, it conforms to the general picture if we regard the single point P on Circle **1** as a one-cusped hypocycloid.

But as well as using hypocycloids inside the rolling circles, we may just as well use the epicycloids outside the circles. The same argument as before shows that L is generated as an envelope by an n-cusped epicycloid escribed to a circle radius n which rolls on the opposite side of the original shaded curve.

These relations are by now a little complicated to visualise, and the challenge is to produce a model which displays them. The most elegant demonstration arises if we merely roll the circles round one another and see how any one of the whole family of epi- and hypocycloids can be generated by all the rest, by regarding the circle to which it is attached as "the shaded curve" and rolling the other circles of the family on it in turn. A theorem may be enunciated in the following terms:—

Given a set of circles, radii 1, 2, 3 . . . where the circle of radius n has an n-cusped hypocycloid inscribed in it and an n-cusped epicycloid escribed to it, if any pair of the family are rolled touching internally, the hypocycloids slide over one another and the epicycloids do the same, while if they are rolled touching externally, the hypocycloid on one slides over the epicycloid on the other and vice versa.

The problem is to make a model which shows this. I gave the problem to Mr. Birtwistle and he describes his solution below.

The Model

The problem was passed to me when I was showing Mr. Fletcher around my Mathematics Department and showed him a Perspex model consisting of a disc rotating in a circle of twice its diameter (one particular case of the general model). This model was roughly made, and as Mr. Fletcher outlined his ideas it became clear that something far more accurate was required. It was essential to have some arrangement whereby one circle could roll either inside or outside a second one, and the model had to be made in plastic since it was necessary to see one curve beneath another. It was decided to arrange this by using a Perspex annulus with a circular disc the same size as the hole offset above it (shown in section in Fig III); by rolling another circular disc around the outside of the fixed disc epicyclic motion could be obtained, while hypocyclic motion was obtained by turning the model over so that the disc was now below the annulus and rolling the second disc around the inside of the hole.

Fig. III

Two difficulties presented themselves. The first was the cutting of the material with sufficient accuracy. My original model had been cut by saw and filed to shape, but this was not good enough. I consulted my colleagues in the engineering department of Nelson Secondary Technical School, provided them with some quarter-inch Perspex and awaited results. They tried various methods, including a lathe, but eventually found the only effective method was to use the circular table on a milling machine and cut the material by means of a cutter in the vertical head. This method enabled the discs to be finished with a tolerance of ·0005 of an inch; this proved particularly important with the modified model ('Mark II' to be described later).

The second difficulty arose when starting to assemble the model. It had been decided to use coloured acetate sheet to show the epicyclic and hypocyclic curves, and to fix this face to face with the Perspex. Also the disc was to be fixed to one side of some clear acetate sheet, while the annulus was to be fixed to the other, thereby getting the 'offset' effect. I soon learned the plastic worker's first commandment—"Never mix your plastics!" It is well nigh impossible to fasten together two plastics with different chemical bases. I did eventually manage to make these stick, although the actual bonding process spoilt the look of the model. It was only by approaching I.C.I. Plastics Division and asking their advice, that any degree of success was achieved. However, it was clear that they could not recommend mixing plastics in this way, and as they are the manufacturers of Perspex, I quickly took up their invitation to visit them and discuss my problem with them. The visit proved very helpful, and as a result of it, I set about making 'Mark II', the improved model.

The main difference here was that by discarding the use of acetate sheet it became necessary to engrave the cyclic curves on the Perspex and at the same time bond the disc and annulus almost edge to edge. The engraving process was done

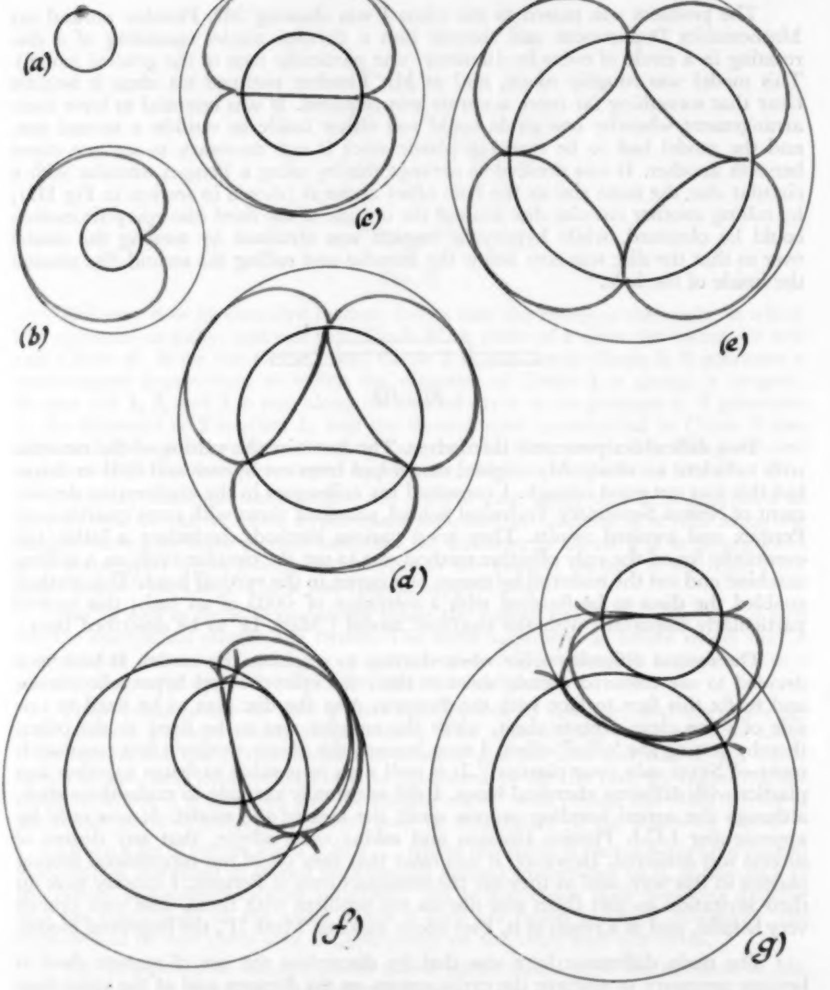

Fig. IV. Generating cirles are in heavier lines.

with a hand cutting tool, and the cuts were filled in with coloured Perspex cement—hypocycloids orange, epicycloids blue. The bonding of the two pieces almost edge to edge meant cutting the disc and the hole in the annulus with a high degree of accuracy so that they made a tight fit; they were then cemented with Perspex cement. The actual model consists of five parts (see Figure IV), each of which is capable of being rolled round any of the others. A small hole in the centre of each part allowed a pencil point to be inserted so that this rolling effect could be achieved. However, as there is a very high velocity ratio in some cases, there is a tendency for slipping to occur. This may be overcome by coating the edges with resinous material or giving them a slight covering of rubber solution.

In use one may start with the circle of unit radius, (a) in Fig. IV, and by rolling inside and outside the other circles, obtain the cycloid curves as the locus of the marked point on the unit circle. This illustrates the well-known method of generating these curves.

However, the model enables a further development to show the property required by Mr. Fletcher. One may take any two of the other parts of the model and rotate the generating circle of one either inside or outside the generating circle of the other. For example, taking the circle of radius two units [(c) in Fig. IV] and rotating it inside the circle of 4 unit-radius [(e) of Fig IV], we find that the hypocycloid of (c)—the straight line—envelopes the hypocycloid of (e), i.e. it is always tangential to it. At the same time the epicycloid of (c)—the nephroid—touches continuously (i.e. has a common tangent with) the epicycloid of (e). [See Fig. IV (f)]

If one now turns part (e) over on to its reverse side, it is possible to rotate (c) around the outside of the generating circle of (e). In this case the epicycloid of (c) touches continuously the hypocycloid of (e), and the hypocycloid of (c) touches continuously the epicycloid of (e) [See Fig. IV (g)]. Similar properties may be illustrated by taking any two parts of the model and rotating them in a similar manner. [In Fig. IV (f) and (g) cycloids only partly drawn for clarity.]

(I must express my thanks for help received from Mr. H. Crossley and others of the Engineering Department of Nelson Secondary Technical School, and to Mr. P. J. Whittingham of I.C.I. Plastics Division, Darwen, Lancs.).

Postcript (by T. J. F.)

These results are not to be found in any of the geometry books to which I normally refer, but they have obviously been known for some hundreds of years, as they underlie the construction of epicyclic gears. The applications of these curves to gears was pointed out by Desargues round about 1625. If gear wheels are cut with teeth whose profiles are based on epicycloids related to the same fundamental pitch circle, then the gears will mesh. Nowadays epicyclic gears have been superseded by involute gears and their practical importance is not so great.

But the relevance of this model to the teaching of geometry lies not so much in the fact that it illustrates a theorem which teachers do not normally teach, as in its approach to the subject. Motion was a concept which did not form part of traditional Greek geometry, and many purists have deliberately exluded it from geometry ever since. But if the idea of motion is cultivated in the study of elementary geometry,

many otherwise separate and dissociated results can be studied simultaneously. In addition, living as we do in an age of machinery, with mechanical movements all around us, a geometry of movement makes more appeal to our pupils' experience than a static geometry does, and it can be seen to have far more application in the world in which we live. The content of the traditional syllabus should be re-phrased in terms of a comparatively small number of mechanical movements. Many of these have been studied in a somewhat isolated way by different members of our Association, and surely the time has come to integrate this knowledge and to remake a geometry syllabus around a small number of carefully selected pieces of mechanical apparatus, and to teach a geometry of movement.

The power of this manner of thinking should not be under-estimated; for consider only the present context. The results stated in this article, including the main theorem are, I find, new to a great many very well educated mathematicians; and if they were asked for a proof they would not find it easy to give one on traditional static lines. But we have indicated how all of these results may be proved, using the model as notation, without the use of a single equation and without the use of any algebra at all, taking as point of departure a quite simple theorem which occurs early on in the school course of dynamics.

[Footnote: many particular cases of the general result described here are illustrated in films by Nicolet, Fairthorne & Salt, and Fletcher.]

TACTILE APPARATUS IN OPERATIONS WITH SIGNED NUMBERS

JOHN W. HESSELGREAVES

At the stage when the pupil is ready to begin work with signed numbers, the plus and minus signs will probably only have been needed as the imperative forms of the verbs 'to add' and 'to subtract'. The signs may not have been used as adjectives to denote 'positive' and 'negative'. On the other hand, the signs suggest to the teacher such varied operations as right or left position and movement, position on a thermometer scale, directional angle measurement. Many of these uses involve the adjectival form, and the dual use is so familiar to the mathematician that he tends to overlook the difficulty to a pupil.

Much progress may be made in elementary algebra before there is any need for signed numbers. The familiar processes and ways of setting out arithmetical processes may be translated into forms in which letters are used in place of numbers. We may introduce the reading, use and invention of a formula. Provided that we select the examples carefully we may make a start with fractions, equations, powers, substitutions, graphs, proportion, leaving aside signed numbers for the time being.

The two methods suggested below show ways in which tactile methods may be used in teaching signed number in such a manner that nothing will need to be unlearned later and there will be no need to impose arbitrary rules of operation. Both methods depend on a double assumption; firstly that in the early stages we may with advantage avoid the dual use of the signs and secondly that is preferable to hold to the usage of + and − as adjectives and to find alternatives for the imperative forms 'add' and 'subtract'. The first method, which is regarded as being the fundamental method, requires an algebraic definition of zero, and defines a negative number as being something which yields zero when collected with the corresponding positive number. The second method needs a positional definition of zero, followed by the concept of negative number as being obtained by the prolongation through zero of a descending sequence of natural or positive numbers.

In both methods the plus and minus signs become labels attached to numbers to show their opposite 'natures', which may be called 'positive' and 'negative'. Instead of 'Add' we will use 'Collect'. To give clarity initially, we will state our exercises in subtraction as 'From A subtract B'.

The coloured cards method

Apparatus. The pupil is provided with a piece of white card 15 inches by 1 inch. This should be marked out into lengths of 1, 2, 3, 4, 5 inches and cut up. A similar piece of coloured card is provided, marked out and cut up.

Nature. The white and coloured cards are said to be of opposite 'nature'. It is convenient to mark the white cards with a + sign and to say that their nature is positive. The coloured cards bear a − sign, their nature is negative. Each of the ten pieces of card should bear its nature and unit value on both sides.

Collecting a, b, c

Collecting numbers of like nature. As we are refraining from the use of the + sign

as imperative, the exercises will be set as *Collect* $+3$, $+2$, $+4$; *Collect* -1, -5, -2. If the collection is performed by setting the numbers down vertically no variation from previous practice is needed. Horizontal manipulation might be set down *Collecting* $+3$, $+2$, $+4$, *gives* $+9$. This is illustrated by setting the cards end to end, in domino fashion. The pupil might now construct a 25 cell collection table with positive numbers from 1 to 5 in random order, following this by a similar table with negative numbers. Later, when collecting several cards of unlike natures, the preliminary assembly of those of like nature is a help.

Collection of two numbers of unlike natures. Here, for the first time, we need to be precise about the manner in which the two natures are 'opposite', and this in turn needs a definition of zero. From the point of view of the analyst, zero may be defined as satisfying the equation $N + 0 = N$, or Collecting N with 0 gives N. The pupil however will readily accept that zero is the result of the operation 'From N take N'. We can then define $-N$ as a number which satisfies the statement 'Collecting $-N$ with N gives zero', and this enables us to proceed.

In the horizontal layout, collecting two numbers of unlike nature will be set down

Collect $+4$, -4	*Collect* $+5$, -3	*Collect* $+3$, -5
Collect -4, $+4$	*Collect* -3, $+5$	*Collect* -5, $+3$

The first of these is illustrated by laying card -4 over card $+4$.
The fourth example is illustrated by laying card $+4$ over card -4.
The two cards taken together, perhaps held by a paper clip, are said to collect to zero, or perhaps to cancel. Other exercises in collecting two cards of opposite natures are illustrated by laying the shorter card over the longer card. After a little practice with single examples, the pupil may be asked to construct a 121 cell table, containing the numbers -5 .. 0 .. $+5$ in random order across the top and down the side, and set in the results of collecting the numbers which lie above and to the left of each cell. The commutative property of 'collecting' may need to be pointed out to the weaker pupils.

Collecting four or five numbers of unlike natures. The pupil will probably discover the advantage of first assembling cards of like natures. In the horizontal arrangement, using $=$ for the words 'is the same as' *Collecting* $+5$, -3, $+2$, -1, $+4$ $=$ *Collecting* $+11$, -4 $= +7$.

Extension to literal quantities. A natural extension of the inch units, using the idea of scale, is perhaps to miles e.g., *Collect* $+5m$, $-2m$, $+3m$. Then in order to bring in different letters and to avoid mixed unit suggestions from the pupils, use pounds money, tons, years. The concepts of above or below the average are useful in this context. One could then proceed to letters in the abstract, N, n, a, b, c, x, y, z using one letter at a time.

Quantities involving two or three different letters. The collection may be illustrated by the provision of further sets of cards of a different width, say 3/4 and 5/4 of an inch.

From a subtract b

Subtraction. We have the additional complication that the way in which the numbers are set down in the question, is not commutative. To give complete clarity the problem might first be stated *From* $+5$ *subtract* -3 although after a time the

pupil will be able to cope with *Subtract* -3 *from* $+5$. The vertical arrangement will need a heading 'In all problems we are subtracting the lower line'.

Any process we devise will need to serve equally for the twelve cases below.

	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	-3	-3	-3	-3	-3	-3
	$+3$	-3	$+2$	-2	$+5$	-5	$+3$	-3	$+2$	-2	$+5$	-5
Answers.	0	$+6$	$+1$	$+5$	-2	$+8$	-6	0	-5	-1	-8	$+2$

As will be seen in the samples below, there is more than one way of using the cards.

Fourth question. With the $+3$, set the zero pair $+2$ and -2. Subtract the -2. Horizontal notation. From $+3$ take -2 = From $+3$, $+2$, -2 take -2 = $+5$.

Fifth question. With the $+3$, set the zero pair $+2$ and -2. Replace the $+3$ and $+2$ by $+5$. Subtract the $+5$. Horizontal notation. From $+3$ take $+5$ = From $+3$, $+2$, -2 take $+5$ = From $+5$, -2 take $+5$ = -2.

Twelfth question. With the -3 set the zero pair -2 and $+2$. Replace -3 and -2 by -5. Subtract the -5. Horizontal notation. From -3 take -5 = From -3, -2, $+2$ take -5 = From -5, $+2$ take -5 = $+2$.

The better pupils will probably find for themselves that the process of setting in a zero pair always succeeds, provided that the numbers are those shown by the second line. But in a number of the examples, this is not the only way.

The slide rule method

This method depends on the recognition that integers, starting say with $+6$, and decreasing by one unit at a time, have a descending order. In preliminary work the plotting of points in the first quadrant, markings on a thermometer, staircase ideas, lift positions in a building with several floors below ground level, pupils' marks above and below the class average may all be used. Formal plotting of points involving the second quadrant would be useful, particularly groups which form a familiar geometrical figure.

Apparatus. The pupil is asked to prepare a line about 10 inches long at the head of a double page in the exercise book, and to mark off in half inch units from the fold. The fold is labelled zero, and the marks are set positive to the right and negative to the left. This scaled line, corresponding to the stock of a slide rule, may be called scale A. A card scale B or slide is then prepared, a convenient length being seven inches. Scale B has a central zero. Both scales may be coloured to the left of the zero, so as to reflect the practice of the coloured cards method. A heavy three inch oval nail would be a useful cursor.

Collecting two numbers of unlike natures. Collect $+5$, -3. Set the cursor on $+5$ of scale A. Set the zero of scale B under the cursor. Move the cursor to -3 on scale B. The result is shown by the cursor position on scale A.

Usage for subtraction. From $+5$ subtract -3. Set the cursor on $+5$ of scale A. Set -3 of scale B under the cursor. Move the cursor to zero on scale B. The result is shown by the cursor position on scale A.

Both processes will serve equally well for any two given signed numbers, and the processes are exactly the same as the pupils may need later in using logarithmic slide rules for multiplication or division.

The question as to whether one or both methods are used in the same year will depend on the age and ability of the class. It is perhaps best to take the development in a leisurely fashion, with other algebraic processes in between the signed number lessons. With a class of age say 12 years and of average ability, eight or ten lessons at intervals of a week might suffice. With older pupils who have failed to grasp other approaches to signed number, perhaps half this number of lessons would be sufficient.

The transition from the vertical layout suggested above to the textbook style is easy; in most cases we will only need to supply an initial + sign where this has been omitted in the textbook. The textbook horizontal layout requires, however, the dual interpretation of the signs. Confusion can be avoided by using coloured chalk or pencil for the imperative form along with the usual practice of enclosing the quantity and its qualifying sign in brackets. The pupil can now be given a little practice in the equivalent styles set below.

Collection or addition.	*Subtraction*
Collect $+5$, $+3$, -4 . Answer $+4$.	From $+5$ subtract -3 . Answer $+8$.
Collect $(+5)$, $(+3)$, (-4). Answer $+4$.	From $(+5)$ take (-3) . Answer $+8$.
$(+5) + (+3) + (-4) = +4$	$(+5) - (-3) = +8$.

From this point the pupil can move to the textbook exercises in the processes, and to their usage in the more elaborate techniques of 'long' multiplication and 'long' division.

AREA AND VOLUME

"Area and volume of everyday objects" it said on the syllabus, and while casting my mind around for suitable everyday objects to measure in the classroom, the thought struck me that we usually ignored the most common "everyday object" —ourselves! What *was* the area and volume of the human body? Not that I had any intention of working them out in the classroom; doing an Archimedes' act in front of the class seemed to be taking practical methods a little too far! Still, I felt we ought to know what the figures were, and discussing how they could be obtained proved a useful class exercise. Discovering references which gave the actual figures proved far more difficult than I had imagined.

I suppose the obvious method of finding the volume is by immersion in a bath. How to find the area is a much more difficult proposition. I remembered hearing some years ago of some medical students doing this by cutting a large number of inch squares of paper and sticking them over the human body, but whether this is the usual method or not I am unable to say.

I could not find a reference giving the volume of the human body, but I found the specific gravity, and this enabled me to calculate that the volume of the average male is about 2·6 cubic feet. A book gave me the figure of 18·3 square feet as an average one for the surface area of an adult, but the scientific reference department of Burnley Public Library produced for me Dubois' formula, which enables the area to be calculated in individual cases:

$S = \cdot007184 \times (\text{weight in kg.})^{0\cdot425} \times (\text{height in cm.})^{0\cdot725}$ where S is the area in square metres.

C.B.

STERN STRUCTURAL ARITHMETIC APPARATUS
(including use with backward seven-year olds).

PATRICIA H. ELLIOTT

As long ago as the late 1920s, Dr. Catherine Stern was concerned about the lack of success of the methods used in teaching arithmetic in the U.S.A. Neither the 'formal system', in which the children were required to do complicated calculations and to memorise both the arithmetical facts and the methods of doing them, nor the 'environmental system', where the children were required to see each number situation against (or in spite of) a background of reality, seemed to be really adequate. In her words, 'so called life situations do not give the child the tools to deal comprehendingly with quantitative relations. Each new situation illustrates one isolated number fact. There is no transfer from one number experience to another.'[1] There seemed to be a need for a system which extracted the relationships between numbers from their background, making them plain to the children, and then went on to show how the number system itself was made up, and how the notation and the formal layout of arithmetic were related to the realities of the quantities involved. Such a system would enable children both to understand arithmetic and use it as an efficient tool.

As the result of twenty years of teaching and research Dr. Stern's book *Children Discover Arithmetic*, was published in 1948, containing the theory and practice of her 'Structural Method' of teaching arithmetic. She claims that her materials have all the properties of real numbers.

Dr. Stern makes use of all the various ways in which numbers can be appreciated. At first she uses two abilities which are developed in young children but seem to get lost later on in life:

1. The ability to recognise patterns and small groups of objects without counting them. This ability is used by the Pattern Boards (Fig. 2).

2. The ability to estimate lengths by eye, and to recognise wholes and compare them, long before their composition from smaller unit parts can be appreciated. (This ability is also used in teaching reading when whole words are learned before the letters from which they are made). The Counting Board (Fig. 1) and the blocks and Number Cases (Figs. 4 and 5) utilise this second ability.

Counting is not used in the early stages of the method. The habit of counting tends to focus the attention on the parts, so that the relationships between the wholes are not noticed; thus children may do many addition sums by counting, and arrive at the answer without understanding the relationship between the parts and the total, and errors in counting are not immediately obvious. Counting is not used in the Stern method until the relationships between the wholes are recognised.

The use of number names and symbols is purposely postponed until this stage has been reached, in order to prevent rote learning of number facts, which may not necessarily be based on any real understanding. Counting is now of use in naming the blocks and patterns and in studying their composition from the units.

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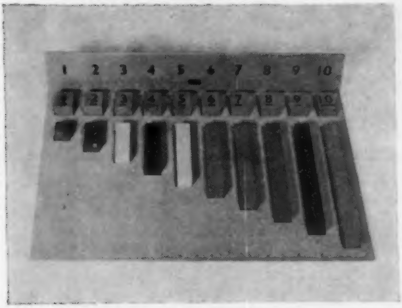
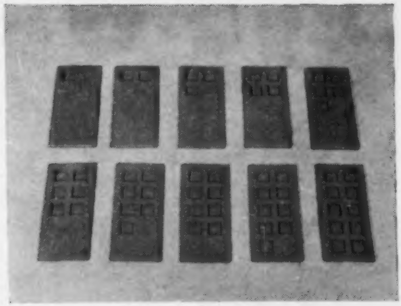
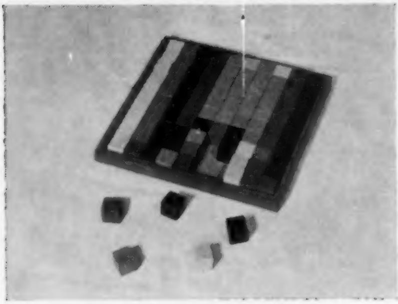
1. Stern: *Children Discover Arithmetic* (publ. Harrap), p.4







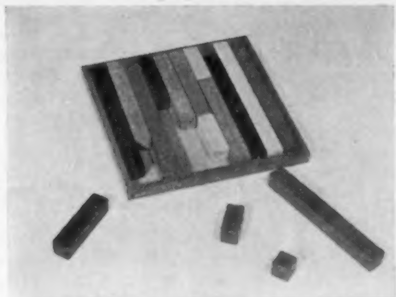


Fig. 1: Counting Board

Fig. 2: Pattern Boards

Fig. 3: Unit Cubes

Fig. 4: Unit Case

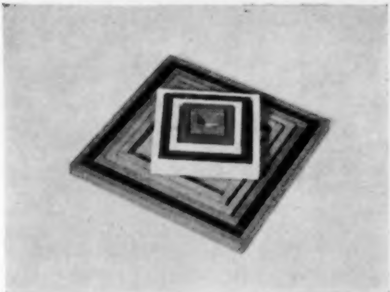
The pieces of apparatus used in the introductory stage are as follows:

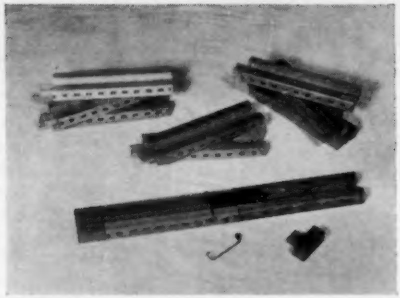
The Counting Board (Fig. 1) has ten grooves to fit the blocks from 1 to 10 in that order. It has detachable number labels and a detachable number guide. The blocks are based on the $\frac{3}{4}$ inch cube and are marked to show the number of units present in each. They are brightly coloured to aid recognition without counting.

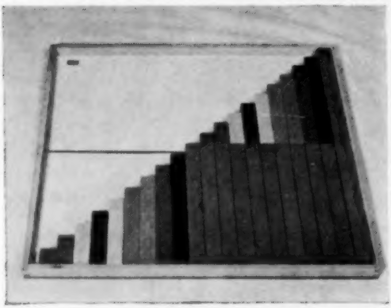
The Pattern Boards (Fig. 2), ten in number, have recessed squares to take $\frac{3}{4}$ inch cubes. The patterns are arranged in Montessori groupings, thus showing that an increase in the number of units is accompanied by a corresponding increase in the overall size of the group. This is not apparent in domino patterns.

The Box of a Hundred Unit Cubes (Fig. 3) for use in the Pattern Boards and Number Track.

The Number Cases (Fig. 5). These are square trays which stack inside one another.







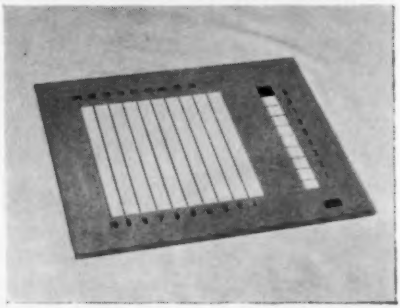


Fig. 5: Number Cases
Fig. 7: Twenty Case

Fig. 6: Number Track
Fig. 8: Dual Board

[*All photographs by courtesy of Educational Supply Association*]

These are used in conjunction with the unit blocks contained in the *Unit Case* (Fig. 4). Each case can be filled with blocks to give the various compositions of that number, e.g. the 'four case' holds the three-block and the one-block, two two-blocks, a one-block and a three-block, and one four-block alone.

At the second stage when the 'teen' numbers and the notation necessary for column arithmetic are being taught, these further pieces of apparatus are used:

The Number Track (Fig. 6) is made up of ten sections, each ten units long, with a trough to hold the blocks. The sections can be jointed together. Their sides are marked from one to a hundred. A wire 'jumper' can be moved up and down the track. The first section of the number track is also used in the early stages for studying numbers from one to ten.

The Twenty Case (Fig. 7). This contains two complete sets of blocks from one to ten, together with ten extra ten-blocks. Thus numbers from one to twenty can be studied.

The Dual Board (Fig. 8). On the left-hand side of this is a recess to take ten ten-blocks side by side. The columns for these are numbered from 1 to 10 along the bottom, and from 10 to 100 along the top. On the right-hand side there is a groove which is ruled into ten units, but only numbered 1 to 9, the top square being blackened to show that no more than nine is allowed in the unit column. The board is used to show column addition, carrying and borrowing.[2]

Dr. Stern advocates using her materials as an introduction to arithmetic at the nursery or early infant stage. Instances are now given of the way in which she does this with young untaught children, to show how the approach with older backward children must necessarily be different. She uses teacher-controlled experiments (games) which can later be used as group games or as ideas for activities in free play.

(a) *Introducing the Counting Board*

The number markers are removed from the filled Counting Board before it is shown to the children. The blocks are then removed and dealt to the children who are asked to judge their size and to fit them into the empty grooves.

(b) *Introducing the Pattern Boards*

Some empty Pattern Boards, say from one to four, are placed at random. The teacher constructs a pattern of say three cubes and the children have to indicate the right board for a pattern of that shape. The number of boards is later increased to six, and then, by even numbers first, to ten. The remaining odd patterns are brought in later. There are several variations of this. The boards are dealt out and each child may be asked to memorise the pattern on his board, without counting it. He then hides it away and has to claim his pattern when he sees it constructed on the tray, by the teacher or another child.

(c) *Introducing the Ten Case and the Blocks*

The full Ten Case is shown to the children and then the blocks are removed and scattered. The teacher places any block in the case and then supplies another block to complete the ten. Another block is inserted and the child is asked to follow the teacher's example and complete the ten, then insert another block and hand the case to the next child, and so on until the case is full. This game is called 'Pass it on'. In the 'Sky-scraper Game' the bottom of the case is filled with blocks of random sizes and the children have a multiple choice. The 'Staircase Game' and the 'Twin Game' are completion games whose names explain themselves.

These games are next played in all the other number cases, and not until the children recognise the patterns and match the blocks accurately are the number names taught.

Teaching Number Names

The cubes are fitted into the Counting Board and counted to give names to the grooves. When these names are known, the blocks that fit into the grooves can be named from them. The Pattern Boards are named by counting the cubes that fill them. Now the number stories from one to ten can be told orally. Next the

2. *Children Discover Arithmetic*, Ch. XVI.

number symbols are fitted into the counting board and the blocks are named. The stories of the numbers can now be written down.

Applying the material to backward seven-year olds.

My problem has been different in dealing with backward children who are already familiar with the number names and symbols. They have been taught at least to set out addition and subtraction problems in equation form. Some can 'perform' carrying and borrowing. The majority however have not arrived at Piaget's Stage Three of understanding.[3] They have no idea of equivalence or of the reversibility of processes. The knowledge that $3 + 4 = 7$ does not help them to do $4 + 3$ or $7 - 4$. If they cannot recall number bonds they guess wildly or count figures or draw dots. Only a few understand the significance of the 'tens column' or the 'units column'. Some will cheerfully total all the digits in both columns!

It is necessary to achieve the following things:

1. To work towards Stage III understanding.[4]

2. To enable the child to think in terms of the language and symbols of arithmetic and to recognise mathematical situations.

3. To make clear the structure of our number system and the conventions of our notation.

4. To give a sound working knowledge of the tools of arithmetic, i.e. number bonds and the composition of tables, so that these facts may be used in quick and accurate computation.

I have used Stern apparatus to try to do this. First, I stop doing any formal writings at all, since in these, symbols give directions for something to be arrived at. I try to make the children see relationships before they do any recording. This cuts out all the over-anxiety about getting answers right. The games I have already mentioned are used to introduce the three main items of the apparatus. The children have to be encouraged to look for whole blocks and not count the parts. Soon the children are ready to play group games as part of their arithmetic lessons. In their free time they can use the apparatus in any way they like; for building, for example, where the relationships between the lengths of various blocks are important.

Although I do not name the blocks at first, the children often find out for themselves from the relative positions of the blocks in the Counting Board, or from the Stair Game or by counting the sections of the blocks. It does not seem to matter whether the 'yellow block' or the 'five' is the name given. The important thing is that the right-sized block to fill the gap should be estimated and selected and *not* counted. The emphasis must be on the *size* of the block. To prevent associations that two colours invariably go together, e.g. the red six and the brown four in the Ten Case, we quickly go from filling the Ten Case to filling the other Number Cases. This is when we play 'Pass it on' as a group game with a child as leader.

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3. See article by Dodwell in *Mathematics Teaching*, No. 5, p. 4 and article by Wheeler in *Mathematics Teaching*, No. 14, p. 30.

4. Laurence Ives, in an unpublished M.Ed. Thesis, has found that in general 'backward children using Stern apparatus arrive at Stage III understanding sooner than those who used unstructured material'.

I find it best to lead the use of Pattern Boards myself, as backward children are inveterate counters and will not look at the pattern shapes at first; they are too concerned with counting. By using the Pattern Boards like flash cards, allowing only a quick glance at a board, then hiding it, and then seeing who can recognise the pattern group made in cubes on the table, or who can construct it for me, the children can be encouraged to look at the whole pattern groups. Then they can each be given a Pattern Board to glance at and hide away while they wait to claim their own pattern when I make it. With very dull children it is necessary to keep to small patterns for a long time. When these are known the composite patterns can be taught.[5] The construction of groups from sub-patterns, e.g. six of one colour and four in another colour on the Ten Board, is something that the children take a long time to do for themselves. They tend to scatter coloured cubes one by one indiscriminately in the boards. So free play with the boards is not very profitable in the early stages, although recognising patterns in the boards, with my help, is useful. I find more use for the boards at this early stage than Mrs. Pleugel suggests in her booklet 'Discovering Arithmetic'.[6]

When the children can recognise the patterns and estimate the blocks needed to fill the cases they can go on to games involving equivalence and reversibility. They fill each groove of the Counting Board with cubes, thus seeing how many units make up each block. They measure the blocks in the first section of the Number Track and substitute cubes for them. They play the 'Twin Game' and then invert one of the pairs to show that the total of two blocks is the same irrespective of their arrangement. This is the 'Upside-down Twin Game'. 'Draw a Number' is a game to show the relationship between patterns and the blocks. The child draws any number from 1 to 10 from a pack of cards placed face downwards. He then chooses the correct Pattern Board and also the block for that number from the Counting Board. He fills the Pattern Board with two colour groups. He then states the sum that the two colours make and then takes out the cubes and arranges them alongside the block. He can thus see that eight is always eight and so on irrespective of the arrangement of the parts.

As the children know the number names they can record their answers. Since these are invariably right, the children gain confidence. They are writing down what they see to be true.

When the number bonds up to six are known, the children can shop up to 6d. in order to put these facts into practice. Change is given by counting on, and a 3d. top plus 3d. change is seen to equal 6d. Very dull children may take some time to see the connection between the facts learned with the apparatus and those in the shop, so we arrange the coins in Montessori groups and make sub-groups of them. When the parts of ten are known we shop with a ten-shilling note for articles costing multiples of a shilling.

Subtraction is first introduced in missing number games in the Number Cases. 'Seven and how many make ten?', etc. Next we take a block and hide away part of

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5. *Children Discover Arithmetic*, pp. 21, 30.

6. W. H. Pleuger, *Discovering Arithmetic* (publ. E.S.A.)

it: 'Six hide away four leaves two showing'.[7] The Pattern Boards are now used. One sub-group can be removed from the filled board and the working recorded. Now to make the relationship between the processes of addition and subtraction clear we take two blocks and make four writings of them:

$$4 + 5 = 9 \qquad\qquad 9 - 5 = 4$$
$$5 + 4 = 9 \qquad\qquad 9 - 4 = 5$$

When the dullest children try to record subtraction after one block has been removed from a filled Number Case, they sometimes tend to focus on one block only and say $6 - 6 = 4$, instead of appreciating that the whole system is ten. This can usually be avoided by suggesting that a finger is drawn down the whole column and the length of the whole stated. If errors are still made the child is clearly not ready for recording subtraction, as he does not appreciate the relation of the whole to its parts.

To introduce more varied language, we place two unequal blocks side by side and say 'What is the difference between five and three?' This can be seen at a glance. Larger differences can be measured by fitting in the appropriate missing block if the answer is not remembered from work in addition.

Games involving addition and subtraction can be done in the first section of the number track, or on card tracks made from $\frac{3}{4}$ inch-squared cardboard. Dice or piles of cards marked +1, −1, +2, −2, +0, −0, are used. Each child has its own track and a pile of cubes. They start with five in already, and the first to reach ten wins.

Now we do shopping for 6d. and for 10s. again, this time recording it as subtraction in equation form. Doubtful answers can always be checked against the blocks and the transaction repeated.

To revise number bonds we play at taking the one-block or the two-block up the stairs. This makes it clear that adding on one gives the next number, and that adding on two gives the next number but one.

When the compositions of numbers up to ten are known in addition and subtraction, and the children understand reversibility and the invariance of numbers, they are ready to go on to numbers beyond ten. About this time we also do multiplication and sharing games with plates and cakes and so on, using numbers up to twelve or fifteen, and following directions on cards to help with reading and carrying out instructions. Sometimes we share and sometimes we put groups together to show the reversibility of the two processes. If the objects we are using are grouped into patterns like the Pattern Boards, errors due to counting can be seen. We use the Pattern Boards for showing groups of two and for halving. Odd and even numbers and remainders of one can be shown, too. We make a square of twelve, by lining up all the possible compositions against a ten-block plus a two-block, then we use the facts in shopping up to a shilling.

The 'teen' numbers can be introduced in the Twenty Case, by constructing the twenty stair and naming the blocks in ascending order; the children see that fourteen

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7. We have some cardboard number shields from America. They are coloured like the blocks and each is used as a hood to cover part of a block. They can be made easily from $\frac{3}{4}$ inch-squared cardboard.

is ten and four and so on. When making the stair we always keep the blocks of ten on the left and the units stair on the right. The composition of twenty in addition and subtraction can be learned. Its likeness to the work in the Tens Case is easily seen by most children. Shopping up to 20s. can now be done for goods costing multiples of a shilling. Writing down the teen numbers is made clear by using the Dual Board, where the tens are always on the left of the units. This is followed by two-column addition without carrying, which the children soon master since they know the number bonds up to ten.

The number bonds from eleven to nineteen need to be taught next before carrying and borrowing can be taught. Adding on to nine may be illustrated in the Twenty Case by substituting ten nines and ten units for the block of ten tens. Each unit-block is then removed and the units block above it drops down one place. It may also be illustrated on the Number Track by, for example, making fifteen from a ten-block and a five-block, then substituting a nine-block and a cube for the ten. Removing the cube shows that $9 + 5$ is one less than $10 + 5$. This gives the idea of adding on to nine, and adding on to other numbers may be shown similarly.

The process of carrying is dealt with on the Dual Board by placing one of the two numbers to be added in the appropriate columns of the board; the other number is put on the table. At first unit numbers to add up to complete tens can be used. Since only nine can be put in the units column, we change the units for a whole ten and carry it to join the tens already in the tens column. This is followed by numbers such as $28 + 4$ and $28 + 14$ where only a small number of units is left over in the units column. The idea seems to be grasped quickly without making the common mistake of carrying the units instead of the tens.

Borrowing is first taught from complete numbers of tens (e.g. $50 - 8$) and later from numbers composed of tens and units. We use the method of equal additions. We subtract the units from ten and then add on the extra units. Dr. Stern uses decomposition and subtracts the units directly from the teen number, which involves bridging. Using the blocks I find it just as easy to explain that we are really taking away an extra ten 'as well as the tens we have already been asked to take away in the bottom line, so we write one more ten at the bottom' ('add a ten to the bottom'), as it would be to explain that we are 'borrowing' a ten from the top of the tens-column and giving it to the units-column. The blocks make the reality of the process clear and it is only the form of words that varies.

When we come to multiplication we can use the blocks to show that three four-blocks measure the same as four three-blocks and so on. This can be done inside or outside the Number Track and the reversibility of the process is easily seen. We build tables in the Number Track, placing cards at the side to mark the 'peaks' of 10, 20, 30, etc. These cards have the names in figures on one side and in words on the other. The number of twos, threes, or fours in twelve, twenty or twenty-one can easily be seen by putting the 'jumper' on twelve, twenty or twenty-one in the Number Track and seeing how many smaller blocks measure up to it, and what space is left over. At this stage it is easy for the children to make the transition from apparatus to written work.

I have not gone beyond this with my 7–8 year olds, but any uncertainties about work later on can easily be explained by reference to the blocks and new processes can be introduced in terms of the blocks.

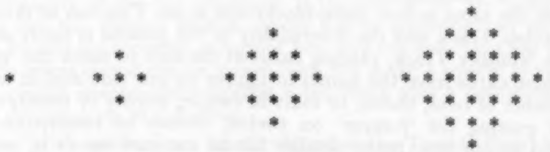
There have been various criticisms of Dr. Stern's method. One is that it leads to S.R. (stimulus-response) learning, e.g. the child learns that *red* + *brown* = *dark blue* instead of 6 + 4 = 10. This might arise if the children were allowed to use the ten Number Case to the exclusion of the other materials, and did not have sufficient opportunity to see the composition of other Number Cases, or to see the numbers composed of cubes of any colour in the Pattern Board, or have the chance to make up big numbers from several small blocks of different colours in the Counting Board or Number Track. The chidren must not be restricted to merely filling the gap above a block each time.

Another criticism is that in order to reach a total, backward children *will* count the units marked on the blocks instead of measuring. The blocks should not be used simply to get a quick answer from a problem card. At the early stage problems must be recordings of what the child has done and not recipes to tell him what to do. He must learn to measure two small blocks against the larger blocks until he finds the right one. He should not be required to do set problems until he knows his number combinations. Even if this does take a long time it is worth it. After a term, or even two terms, of very slow progress the ideas suddenly seem to fit together and learning takes place more quickly. One cannot force backward children to learn; one can only put the materials in their hands, and see that they enjoy plenty of meaningful practice. They must not be encouraged to learn the verbal answers before the meaning is there.

I have found that if the intention behind Dr. Stern's apparatus is carried out, with variations to allow for the effect of several years' teaching, the apparatus is a great help in teaching arithmetic to backward children. The teacher must understand the purpose behind the method and not regard it as a short cut to verbal repetition of uncomprehended facts. She must be prepared to wait until the child arrives at an understanding of conservation and equivalence, before going ahead with computation.

PLAIN DOTTY

How many dots are there in the 100th term of the following sequence?



etc., etc.

Answer to 'Fair Deal' in our last issue

The cards are arranged (reading left to right for top to bottom of the pile):
R R B B R B B B R R B R R B B R R R B B

SECONDARY MODERN SCHOOL MATHEMATICS — VI

SOME GEOMETRICAL TOPICS

ARNOLD IVELL

We are all aware of the repeated calls to teachers to thrust aside the accepted traditional approaches to the teaching of mathematics, and to step forth boldly in the school with a new syllabus, and a revolutionary modern approach to our methods. But how many teachers hesitate to answer this call? How many doubt the wisdom of introducing such 'revolution' into our schools? How many consider that in the end, all this business is a mere 'gimmick', soon to be buried and forgotten for new 'gimmicks'?

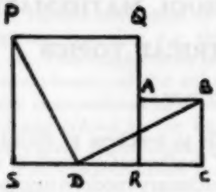
This article is expressly written in the hope that its contents will encourage and stimulate those teachers eager to assess sincerely the merit of this new approach, particularly in the teaching of geometry in the Secondary Modern School. At the same time, it is hoped that this article will reveal to those already convinced, some fresh ideas well worth trying in the classroom.

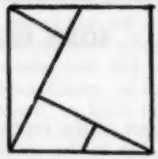
As traditionalists, taught ourselves by traditionalists, it is easy to think of geometry solely as a world of proofs and constructions based on the Euclidean theorems. In the Modern School, some may hesitate to teach any geometry at all on the basis that the pupils' intelligence does not permit us to pursue the subject usefully. They offer in its stead more arithmetic and further practice of the usual fundamentals. "After all," they say, "If Joe doesn't know his tables, why teach him geometry?" Must we learn to speak Chinese before we are permitted to learn of China and its peoples?

Let is be clear from the outset that it is not intended to divorce geometry from the rest of mathematics. Every opportunity will be grasped in this article to make use of as much arithmetic and algebra which naturally occurs as each subject is developed. The author does not wish to analyse here or even suggest a geometry syllabus, but certain items have been selected to show how one can introduce in a fascinating, new way, mathematical principles and data previously denied to our pupils. Let us look at some of the ideas:

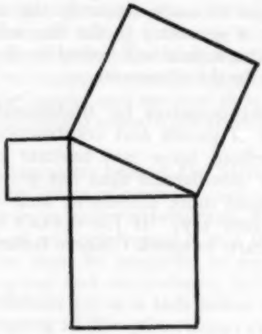
Pythagoras. In introducing Pythagoras's famous theorem, the traditional way has always started with the right-angled triangle and then reached the fact that two squares when added form a third and larger square. Why not break with this tradition and work in reverse? Why not start with two squares and the resultant third square, and show that they lead to the right-angled triangle?

Let any two squares be cut out of paper and set side by side so that their bases are in one straight line. Mark off CD such that CD = PQ. Join DP and DB and cut along these lines to give five pieces that can be rearranged to make a third square. This dissection can be carried out with any two squares of any size. The pupils will thus appreciate that ANY two squares can be added to form another square and not merely those squares of sides of 3 and 4 units.





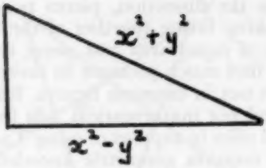
If squares are then cut equal to those squares used in the dissection and for the resultant third square, then there must be some connection between these three squares. It is possible to demonstrate their being connected by arranging them in such a pattern where they touch at their vertices. It can be seen that all such patterns



as those obtained from the pupils' dissections result in one common fact—that there is a right-angled triangle formed by the connected squares. From this it is easy to present the converse: If we start with a right-angled triangle the squares on all three sides are connected with each other such that the largest square is equal to the sum of the two smaller squares.

This is not the only dissection that leads to Pythagoras's theorem, but the Chinese dissection (where one of the squares is cut into four congruent pieces) requires that the right-angled triangle is obtained first for the dissection to be carried out. The "Classical" dissection described leads to, and does not depend upon, the right-angled triangle. One teaches from the known to the unknown, from squares to the triangle of certain geometric properties.

Before leaving this subject, it is of interest to show the class a very fascinating manipulation of numbers that results in the three sides of a right-angled triangle. Take any two numbers. Double their product; subtract their squares; add their squares. This gives three different numbers each being the length of a side of a triangle which must be right-angled. Try any two numbers for yourself and check the result. Amazing, isn't it? Let us examine the general result:



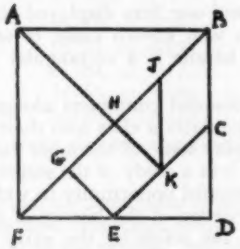
Any two numbers x and y.

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$$

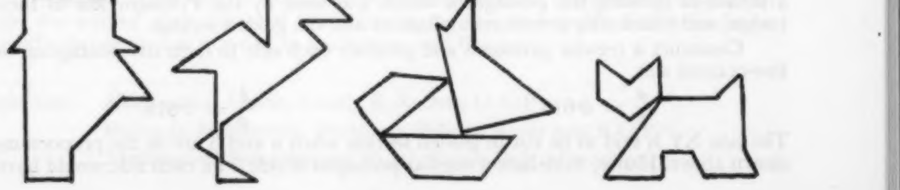
$$\begin{aligned} [\text{Since } (x^2 + y^2)^2 &= x^4 + y^4 + 2x^2y^2 \\ \text{and } (x^2 - y^2)^2 + (2xy)^2 &= x^4 + y^4 - 2x^2y^2 + 4x^2y^2 \\ &= x^4 + y^4 - 2x^2y^2] \end{aligned}$$

Hence, this manipulation will operate for any values of x and y and therefore for any two numbers.

Areas of geometric figures can be studied via dissections, and a fascinating by-product of such dissecting is the subject of Tangrams, studied some 4000 years ago in China.



Tangrams. Draw any square ABDF and dissect it as shown in the diagram such that C is the mid-point of BD, E is the mid-point of DF, and AK is almost the diagonal AD. Using the seven pieces, reform them without omitting any piece or without any overlapping whatsoever, into these (and other) familiar shapes. It is not so easy!



As one can tell from the dissection, pieces possess edges which are equal to edges in other pieces, making fitting together of the various pieces a very puzzling problem. All patterns are of equal area and prove that area depends upon surface and not shape. The class find much pleasure in inventing their own shapes. I have seen a whole orchestra set out in tangram figures. Rouse-Ball states that Tangrams are purely recreational and not mathematical, but from the teaching point of view I would challenge this and offer in support of using Tangrams in school the argument that it gives pleasure, it requires geometric knowledge to dissect the square and introduces the parallelogram, right-angled triangles, congruent triangles and, above all, a practical appreciation of spatial relationships.

Polyhedra. We live in a world of solids, a world of three dimensions, and yet tradition has ruled that we teach plane geometry often to the exclusion of any solid figures. What a false situation to present to any child that the world consists solely of plane figures. It is therefore never too soon to introduce our pupils to the real world in which they live and with which they are familiar from birth, the world of three dimensions.

Boxes of various shapes are easily found and will help show to the pupils everyday examples of the use of simple solids. To demonstrate a few such excellent examples we can collect soup tins and cheese boxes as cylinders, fancy shaped chocolate boxes as examples of hexagonal prisms, and one famous make of chocolate has a container which is a triangular prism and one firm displayed their Easter eggs in truncated hexagonal pyramids, whilst a well known make of salt is packaged in truncated cones. Nearly every box we handle is a rectangular prism whilst some are conveniently perfect cubes.

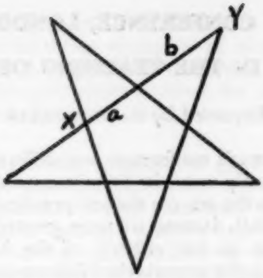
An exhibition of such colourful containers alongside models of their mathematical equivalents always stimulates a class into their wanting to make their own models particularly if you display some of the more exotic shapes. From this desire we can happily and easily go into a study of the simple solids and so introduce the 5 regular solids. This is a wonderful opportunity to widen the pupils' mathematical horizon concerning their knowledge of the history of the subject. Mention can be made of the aesthetic use of the solids by the early Egyptians, how Pythagoras contributed to the mathematics of these solids and Archimedes who extended Pythagoras's studies. One should not exclude how Plato used the solids in his philosophy of life and the world, or overlook modern contributions via the studies of Coxeter.

The work of Archimedes will introduce the simple rules of stellated figures, and will reveal the "golden section" or "divine proportion" as it is also known. The mention of Pythagoras as a contributor to the study of the "Platonic Solids" can be a means of showing the pentagram which was used by the Pythagoreans as their badge, and which also introduces stellations and the golden section.

Construct a regular pentagon and produce each side to form the pentagram or five-pointed star.

$$\frac{a}{b} = 0{\cdot}618 \qquad\qquad \frac{b}{a} = 1{\cdot}618$$

The line XY is said to be cut in golden section when a and b are in the proportions shown above. Hence, to stellate a regular pentagon of side 5 in. each side would have



to be produced by 8·09 in. This information is essential for the stellating of the simple solids and in preparing the nets for their construction.

The golden section gives rise to a simple yet valuable study of the Fibonacci Series, an excellent practical demonstration of the use of ratios, and linked with the geometry being studied. This series is infinite and starts 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, etc. . . . where each term divided by its successor reaches the limit of 0·618 which is the golden mean.

1/1 = 1·000	8/13 = ·615	89/144 = ·6180
1/2 = ·500	13/21 = ·619	144/233 = ·6180
2/3 = ·666	21/34 = ·617	
3/5 = ·600	34/55 = ·6181	
5/8 = ·625	55/89 = ·6179	

Have you discovered how this series can be continued? How did Fibonacci himself discover the series?

If rectangles are drawn so that the length and breadth are in golden section, and then squares are constructed in these rectangles so that the remaining portion of the original rectangle is itself similar to the original, and this construction is continued, a continuous curve from diagonally opposite vertices will result in the logarithmic spiral. Of value to the Modern School child? Of course it is. Nature makes use of this spiral as we discover from the growth of the snail shell and children certainly handle such shells or have at least seen them. It adds to their mathematical background to be taught that this spiral is called a logarithmic spiral and can be obtained as they were shown in their geometry lessons.

These are but a few of the many interesting features in geometry which can make the subject alive, divorced from learning by rote and application to books on Euclid's theorems, but married to fascination and beauty.

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EASTER CONFERENCE, LONDON, 1961

DEVELOPMENTS IN THE TEACHING OF MATHEMATICS

Reported by DAVID FIELKER

Everything about this year's conference was different. Memories of Blackpool soon faded away, with its family aura, its luxurious accommodation, and the doors of the hotel opening out onto the sea (or the sea practically coming into the hotel—the winds are strong in March!). Instead we were greeted with the homely but more sober atmosphere (there was no bar either!) of the National Society's Training College of Domestic Subjects in the unperturbed fallowness and Victorian architecture of its Easter vacation, a cold, calm background to the warm urgency of our convention. Somewhere to the East lay Hampstead Heath in the first fullness of spring, but either it was too far away or we were too busy to care.

If there was any significance in our geographical position then it was firstly that being in London meant that nearly twice as many people attended this year's conference, and secondly that the nations of the world met in the conference hall just as they did in Finchley Road and Haverstock Hill. Questions were asked in a variety of British and foreign accents; greetings were announced from Mr. Fletcher in Canada and Mr. Trivett in the U.S.A., and one of our speakers was Professor Georges Papy from the University of Brussels. We felt, with Mr. Fletcher, that our problems were *world* problems.

Prof. Papy certainly left no doubt about his nationality. He spoke with a strong Belgian accent, submitted with excellent good humour to our confirmation of his choices of pronunciation and delivered the occasional French word with a questioning gesture to the bi-lingual mathematicians in the audience. (Specialization has not conquered us yet!). His vocabulary was strangely unbalanced; mathematics was "omnipresent" but an example of a set was a "flock of muttons"! His humour and wit, usually too integrated a part of his lectures to quote out of context, added to the forcefulness of his presentation. Nobody else, I am sure, has described Euclid (the pronunciation was corrected by the second lecture, but I'm glad he didn't mention Euler) as "the second best seller in the world"! In particular his startling revelations of his own student-day thoughts were amusing as well as illuminating and we quickly gathered the impression that Prof. Papy was as extraordinary a schoolboy as he was a professor. I was not altogether happy about some of his examples, but even these were given with a deceptive clarity that ensured that one investigated them afterwards.

Dr. Ruth Beard, from the University of Birmingham, was more conservative in her approach, but then she was more conservative in her subject-matter. Piaget is established in the world of education; modern mathematics is not, or, at least, not where we would like it to be. Dr. Beard was giving information rather than fighting a battle, and this information she gave with admirable clarity, illuminated by examples quoted from her own researches. It is not often that one has the chance of such authenticity at so high a level.

Mr. Hope we knew. Or rather Mr. Hope we knew some of the time; for whether it was due to the comparative augustness of the occasion, to the importance and

newness of the ideas he was presenting or to his new position as President of the Association, Mr. Hope did two things strange to all of us—he kept his jacket on, and he read from notes! Neither hampered his customary initial greetings and a story too long to repeat, or the frequent desire to leave the table and parade the platform with characteristic gestures. But each time, alas, he interrupted his orations to return to his notes.

Our other speakers were not without the good humour that pervades and enriches the proceedings of the Association. On the first evening Mr. Harris was presented in glorious technicolour while somebody else selected suitable lighting. Mr. Birtwistle, first of a trio of speakers, said the best bat opened in cricket and the worst act went on first in variety, and left the rest to us! Better, he thought, than giving the last talk of the conference; he wouldn't like people to read "Talk by Mr. C. Birtwistle—(tsk!)—'What next?' " Mr. Wheeler, speaking last, said it was difficult to find something that the others hadn't said, "especially as neither said what they said they would say!" Halfway through he confessed to losing his place in his notes. "It sounds like it", said Mr. Beaumont. "Never mind," Mr. Wheeler went on, "I can remember the next bit!" Perhaps one day someone will make a collection of these and other snippets such as the statement "I don't know any educational psychology", which came from the lecturer in Education at Leicester University!

However, all this was but sauce (no pun is intended) for the meat of the conference, perhaps one may say the meat of modern mathematics surrounded by the vegetables of Piaget's educational psychology. For rather than two complementary themes one felt that Dr. Beard's lecture on Piaget's work was a supplement to those of Mr. Hope and especially of Prof. Papy, in the sense that Piaget's experiments showed that modern mathematics was, to put it simply, the correct and more natural thing to teach. In this almost revolutionary atmosphere the seminar groups with their more familiar topics—aids, first year in the secondary modern school, the sixth form, film-making, methods of teaching this and that, syllabus construction—seemed to hark back to another era.

Two main problems, dealt with as far as was immediately possible in the final discussion, emerged in the formal discussion groups and in the conversations that went on all the time. Firstly, we are confined to our present system by examination syllabuses. Will a G.C.E. examiner accept a vector proof of Pythagoras? Will anyone provide questions on set theory in a G.C.E. paper, or even in a Junior Leaving examination? But the second problem must be solved before pressure can be brought to bear in the right quarters to do something about the first. At present (in England and Wales at least) the only teachers in general with a knowledge of modern mathematics are those with fairly recent honours or special degrees. These form a subset of the set of graduate mathematics teachers, which in turn is a subset of the set of all mathematics teachers, which in turn is a subset of the decreasing set of mathematicians mentioned by Mr. Birtwistle. The problem of teaching modern mathematics is thus enclosed in concentric vicious circles. Three lectures do not constitute a comprehensive course in set theory; they can but give, it is hoped, the inspiration to seek for more information. Books are available, but they are not enough. Study groups were formed, but they obviously have limited scope. Surely group and private study must be coupled with research in the classroom. It is known that Mr. Fletcher

is beginning such a project in London in September. Need we wait? Does it matter if we make mistakes at first? Mr. Wheeler certainly thought not; teachers in general seemed more apprehensive.

But the conference as a whole seemed to be a great success. Even apart from the ideas that were being presented formally there was much to talk about. Secondary teachers in particular were impressed by the enthusiasm of their colleagues and found it extremely helpful to exchange views and ideas. One primary teacher confessed that he felt neglected, perhaps not appreciating Piaget or believing that he could teach functions to his ten-year-olds. But mathematical talk went on in the dining room over the food, in the sitting room among the display of structural material, and in the conference hall round the models, the visual aids, the calculating machines and the electronic computor made at Whitgift School, Croydon. Even the two theatre parties ("West Side Story" and "The Devils") returned to the sitting room for tea and discussion, and had to be thrown out with the rest by a tired porter. Mr. D. T. Moore had helped to arrange many things for us, but if nothing else (and there was plenty) he had arranged for nearly two hundred teachers from all over the country to meet each other for three days with a common aim, the improvement of mathematical education. This alone was worth while.

Welcome and Introduction

Mr. I. Harris welcomed us to the Conference, explaining that Mr. Fletcher and Mr. Trivett, who sent their apologies and best wishes, were in Canada and the U.S.A. respectively, and so he was the sole surviving chairman of the Association.

The Association began in 1952 in order to rethink mathematics, and our aims had not changed since then. We had to gain an understanding of children and the ways in which they think. We had to consider mathematics as a whole, and see what results we could obtain from this. Also the Association acted as a clearing-house for ideas in the teaching of mathematics through its bulletin, through local groups and through its film unit.

A letter from Mr. Fletcher emphasised that our problems were *world* problems (he was engaged in making a film whose contributors came from various European and American countries), and other activities with which we involved ourselves must also expand on a world-wide scale. This international flavour Mr. Harris saw as a reflection of our vigour.

He applied a quotation on education from a Sunday newspaper to mathematics in particular: after the eleven-plus, G.C.E. and a degree we had squeezed all the mathematics possible out of our pupils. Was this true?

The Challenge of the Situation

It had been suggested to **Mr. C. Birtwistle** that he dealt with the errors in the present situation, and the general picture of education must come first. Education was not *for* life, but *was* life. Man must be fitted for his environment, morally and socially as a member of the community, in all aspects as a fully developed individual, and vocationally as a contributor to society through his work.

Modern man's dilemma was self-realization. Science had expanded individual horizons, our community was now world-wide, and the ends of the earth were brought nearer by modern communications and by radio and television. But man viewed life around as an observer, not a participant, and he lacked a sense of belonging. The result of recent changes in society was an emphasis on materialistic values; "if you have the lion stamped on you, you're a good egg!" The worker was merely part of a machine, and though the corresponding increase in leisure time ought, in Aristotle's view, to be devoted to "what is good," more leisure had come to mean more self-indulgence, and hence less self-realization. However, the criteria of the past were not those on which to base a solution of the problems of today, any more than today's criteria would do for tomorrow's problems.The vast majority of people had no interest or use for the traditional British liberal education based on the Greek ideal—culture untainted by utility. Useful knowledge was that which had relevance and meaning to life, and in turn such education would *bring* relevance and meaning to life.

We must therefore consider the aspects of twentieth-century society. A technological age demanded an understanding not solely of human but also of scientific relationships and their applications. Increased leisure must be devoted to that individual realization on which the progress of society depended, but education had not inspired this. "Education for all", though existent for a hundred years, was still based on the needs of the intellectual elite, and did not provide for each individual. Independence of thought was necessary for that individual contribution in thought and activity demanded by a democratic society.

In practical terms, general education must bring relevance and meaning to the student and his world, but we must look even more deeply than this. The teacher might achieve this by his methods, but connected with method was syllabus content hide-bound by tradition. Science in particular showed how our teaching contained out-of-date material and ignored the developments of the outside world which meant more to present-day youth. Syllabuses, though, were shackled to external examinations, the criteria of intellectual attainment, and even the once examination-free secondary modern schools were judged for success on their G.C.E. passes. Examinations, concerned mainly with vocational education, had become an end in themselves. Universities in particular sinned here, and perhaps some altogether different institution of education giving training in citizenship and use of leisure would satisfy more needs than would an increase in University places. Were the new three-year training college courses providing a greater breadth of education or merely succumbing to the present preoccupation with specialization?

Mathematical education erred first in the vicious circle of the decreasing supply (in comparison with the demands of science and society) of mathematicians. Specialists were needed, but we must also educate the "numerates" of the Crowther Report, appreciative of the mathematical problems of today. Thus the requirements of education in general applied particularly in the teaching of mathematics, and to achieve those aims we should impart the ideas behind the subject. To be able to do this we must understand the psychology of our pupils, and decide on the order in which the ideas must be presented. *This* order should coordinate the subject, and we should not rely on tradition. *New* thought was required.

(*Continued on p. 38*)

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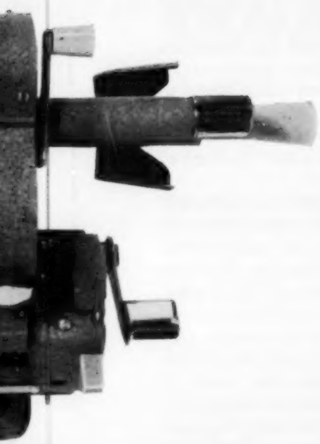

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At the primary stage, understanding was beginning to take the place of mere drill in technique, and it was here that we must lay the foundation of the student's attitude to the subject as well as of the subject itself. The secondary school must adopt a broader approach, for the pupil was actively interested in current developments in his world. In particular the modern school must make use of its freedom, especially as in most cases it represented the end of formal education. The sixth form could provide a breadth of mathematical education even within the restrictions of competitive examinations, and the specialist could be concerned with the historical, aesthetic and cultural aspects of mathematics.

It was to the universities that we looked for a trend in new thought, but any radical thought was choked by tradition. We, as practising teachers, must demand a change, and a release from the restrictive influence of examinations. We must be free to investigate and experiment, and we must have the power to take action wherever we found fault.

Mr. G. Beaumont concerned himself more with the situation in the classroom, and investigated first the distribution of blame in mathematical education, which passed from the universities stage by stage down to the infant school. The shortage of teachers was an overall problem. The problem of teaching was still there, but the problem of shortage, in spite of various experiments to cope with fewer mathematics teachers, would remain till mathematics was really popular. 'O' level mathematics had become a "snip", but was not popular when the first and second form pleasure at receiving "ticks" had passed.

The conference should consider three things—the relevance of concept formation to understanding, the place of modern mathematics in teaching and the use of structural material.

With reference to the second of these, we were asked to consider the different footings of algebra and geometry. In geometry we had freedom, particularly in projective geometry. But algebra, as generalised arithmetic, was rigid and useless. However, there was some relevance in certain points of modern algebra. (Unique factorisation was an example, and some good work could be achieved by a sixth form investigating the differences between modulo 5 and modulo 6 arithmetic).

We need not be despondent, for changes in the teaching of mathematics did take place. (We no longer learnt geometrical theorems by heart! Indeed, proofs could be based on vector algebra). Recent investigations supported the claim of modern mathematics by showing that young children had the embryo ideas of it before they met formal arithmetic. It might well be that it was more natural to go from algebra to arithmetic instead of vice versa, and from topology to Euclidean geometry via projective geometry.

From the demands of society, of mathematics itself and of the teacher-pupil relationship, **Mr. D. H. Wheeler** chose the third to discuss. The first involved more "numerates" (that "dreadful word" from the Crowther Report!); in looking at the second there was a danger of looking only at mathematics; the third meant psychology ("I'm afraid," said Mr. Wheeler apologetically!) and even though we know little about psychology at present we at least could steer a middle course between knowing everything about children and believing them too complex ever to know.

Our teaching had been affected by Piaget, who had shown that a person's *thinking* grew as well as the person. Curiosity came before an urge to explain, and we must develop the type of thinking that dealt with explanation. Verification of opinion was necessary, even in adults, and explanation was necessary in defence against a contrary opinion as well as in straight-forward communication.

When we thought of "five", did we see a written symbol, the dots on a domino, the fingers of one hand or a yellow rod? In order to explain we must first construct models, probably mental ones, which were usually visual. (One could get set in one type of imagery, and only change when that particular model was found to be defective). Then our models must be criticised and improved. We were now on the way to abstract thought, although in order to communicate our abstraction must be put into concrete form by way of an illustration or example.

Another feature of this development of thinking was the progress from complete assurance (the state of being unable to explain) to humility (being able to explain, but knowing that frequently changed ideas may have to be changed yet again). It was also said that adult learning was less efficient than that of children. If this was true, then it was a pity that we taught children who learnt more efficiently and quickly than we did. Hence the importance of humility. "We should have the courage not to understand".

Piaget's Investigation of Children's Mathematical Thinking.

Dr. Ruth Beard assumed that we had some knowledge of Piaget's work, which covered the complete school range from infants to sixteen-year-olds, as could be seen from his books on such topics as conception of number, space and time as well as verbal reasoning, scientific and moral concepts and general psychology. Dr. Beard proposed to discuss the second of the three periods which she first briefly described.

The first or Sensori-Motor period, lasting from birth to eighteen months or two years, saw the organisation of actions in relation to surrounding space, but there was no evidence of imagination or thought as we knew it. Period II was divided into IIA, Preparation for Concrete Operations (2–7 years) and IIB, Concrete Operations. At the beginning of IIA there was evidence of memory, e.g. of other children's actions and where things had been hidden; more thinking was now involved, language was used, and children were able to imitate speech and action; but these things had now to be organised. Hence IIB, during which took place the development of reason and the ability to perceive relationships and classifications. Period III, the period of Formal Operations, was illustrated by an experiment in which children of various ages were asked to obtain a certain colour by mixing liquids; the younger children worked haphazardly and stopped after their first success, but those between fourteen and sixteen worked systematically and were not satisfied with just one solution.

Period IIA could be subdivided into (a) the preconceptual stage and (b) the intuitive stage. During the latter, i.e. from four or five to seven years, children were becoming adapted to their environment. This adaption was a combination of (i) assimilation, incorporating something new in established behaviour, and (ii)

accommodation, adjustment to a new situation. Characteristic was a lack of understanding of quantity, due to a consideration of only one aspect of the problem. When beads were poured from a small to a large box, children tended to say there were more or less beads, according to whether they perceived that the box was larger or the beads were more spread out, and so less dense. When beads were put, bead for bead, in a dish and a U-tube respectively, and children were asked which set would make a longer necklace, they perceived the length or tallness of the tube or merely the fact that it was a tube! In comparing the sizes of rectangles of different shapes children of seven-and-a-half were dominated by the sides that faced them, though one bright six-year-old imagined unit squares and others were able to "fit in" the area.

During period IIB classification became possible in various forms. Children could put such things as Montessori numbers, areas, weights and simple volumes in order, but if there was any complexity then there were limitations; only three things could be dealt with up to eight years and four or five or more at about nine years. Piaget found that conservation of mass was appreciated at six years, weight at eight years, volume between eight and twelve and density between nine and twelve years, though no difference between the first three had been found by a research worker in Aden. A typical experiment was asking children about a ping-pong ball and an equal volume of Plasticine each under water, the former being held just below the surface with a finger. Dr. Beard had found that children between six and seven considered that equal volumes had equal weights.

In experiments concerning perception of part and whole Piaget had asked "Could you be in Geneva and Switzerland at the same time?" and only two-fifths of a group of nine-year-olds had said yes. But Dr Beard had applied the question of Acton and London, and half of some six to seven -year-olds had agreed, some quoting their address to prove it! Perhaps, though, there were differences both in time and educational background.

Most children were able to put a farthing inside or outside a circle, and agreed that a farthing placed across the circumference was not properly inside, thus the meaning of the terms was clear. When confronted with two intersecting circles, a third of the children consulted were able to put the farthing inside one but outside the other, and a third could place it inside both, though there were protests that two farthings were needed, and that "outside" meant outside both circles. When one circle was completely contained in the other, again a third of the children could put the farthing inside the exterior but outside the interior, but there were protests again when they were asked to put it inside both. Since it had been suggested that solids were more familiar, Dr. Beard had tried asking children to put a marble in both of two boxes, one of which could fit inside the other. A seven-year-old boy with a mental age of eleven had protested that "even a king couldn't do that!" (He eventually had success). Some tried tossing the marble quickly from box to box, and some arranged the boxes as in Fig. 1, so that the marble was topologically inside both.

Perspective became evident to children at about the age of nine. Piaget showed a map of three mountains with distinguishing features, and a doll was placed somewhere on the perimeter. Children were asked to choose the doll's view from a set of pictures. The younger children tended to choose their own view, but those above nine chose correctly. Conception of horizontal and vertical became evident in

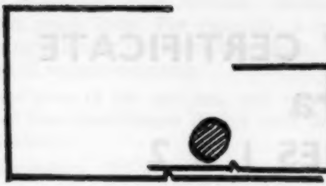


Fig. I

children's drawing at nine, when houses and telegraph poles became vertical and not merely perpendicular to the hill on which they were situated. Children shown pictures of their room upside down and upside down with lateral inversion could distinguish vertical inversion but did not realize till the age of nine that there was also lateral inversion, though in a slum school Dr. Beard had found better success, due, she thought, to the handstands and other gymnastics she saw in the playground.

Piaget had asked children to join two model trees by a straight line of trees, and found that children followed the perimeter of the object on which the trees were placed. However, Dr. Beard thought that the difficulty here was one of terminology. She had presented thirty-nine intelligent eight-year-olds with various lines drawn on a sheet of paper and asked them to show the straight ones. All agreed that vertical and horizontal lines were straight, most a fairly flat curve, none an oblique line, and only six a straight but dotted line.

Besides the question of terminology, there were other criticisms to be made of Piaget's experiments. Were the ages he gave correct? Mental ages seemed to be more important than chronological ages. Mr. Collinson of Birmingham had shown that chronological age was as important as mental age in history; but mathematical experiences were available to children whereas history involved adult experiences unknown to the child.

How important was experience? Two Malayan girls of fifteen and sixteen whose number work was good had reacted like six-year-olds to a conservation of volume experiment involving deformation of Plasticine, possibly because they lacked the experience of English children. An experiment at Leeds was carried out with two groups of five-year-olds, one being exposed to experiences of order. They behaved like seven-year-olds when given a Piaget-type test after a time, and a further test later showed a similar result. Thus it seemed essential to provide correct experiences.

Cultural background seemed to have some effect. In Aden, where white parents were well-off compared with others, white children were found to be more advanced, though they followed the same order of concepts.

Experiments in Birmingham suggested that Piaget's stages existed for blind children, even though they overlapped and were not clearly defined, but evidence was as yet conflicting. Also it seemed that children progressed at different rates in different fields.

Answering questions, Dr. Beard discussed the effect of Piaget's work on primary teaching. It was mainly concerned with method; children needed practical experience before they could grasp ideas. For example, a report had recommended that area should not be taught before the secondary stage, but we had heard earlier of an infant's appreciation of square-counting.

Piaget never gave the sizes of his samples, and whereas Dr. Beard used the same sixty children for all her experiments Piaget used different groups; but then he apparently despised statistics!

Experiments with secondary children seemed to confirm the idea that the progression from topology to projective geometry and then Euclidean geometry, was most natural.

A New Structure for School Mathematics

Mr. Hope commenced by saying that the learning situation had two aspects—the learner (the child) and the subject (mathematics)—and it was the teacher's task to bring the two together in a fruitful union. In considering school mathematics, therefore, we have, on the one hand, to consider a *child centred* scheme of work; it must look outward from behind the child's perceptions—not focused on the pupils but focused *from* the pupils as origin. On the other hand it must be centred on Mathematics.

Our Association has founded its work on the child-centred approach. It was his intention to deal with this aspect, perhaps the most important, only briefly so that we could concentrate on the mathematics we hoped to teach, an aspect which A.T.A.M. had not considered in quite the same detailed way.

A child in his own universe has his own problems. What he sees is often much more than the teacher knows. Very often his perceptions lie outside the realm of traditional school syllabuses and may form the basis for topics of "Modern" Mathematics. The common or garden trellis work may be just a lattice of parallelograms (and Euclidean at that) in the teacher's view, but a normal child may see only the collineations which are invariant with respect to all stretching or closing of the lattice. As such it is a model for a vector algebra and if we accept the absence of any perception of angle as playing a role then we could develop affine geometry from this basis. Perceptions pose problems and it is the teacher's task to thread a way through mathematics guided by the children's problems. The rate at which one may proceed, the complexity of the problems we may study are determined by the pupil's attitudes, his interests, his maturity, and his personality. His growth is dependent mainly on the degree of involvement in what he is doing. These ideas are well known to us; they underlie our approach to teaching aids, to classroom demonstrations and all the rest of it.

The teacher, the link between the learner and mathematics, must be alive to what is going on in the world today and to what is setting the stage of tomorrow. He must have a great depth and width of background, a readiness and quickness to assimilate ideas, new and traditional, for he, too, is a learner. He must have skill in presentation and above all an understanding of his children as mathematicians with frequent contact with individuals in mathematical contexts. Each child is a

genius in his own universe; we must be careful to explore *his* universe with him so that its limits may be extended. Perhaps too many of us are too concerned with doing the mathematics of next year with the children of last year to be aware of the needs of the real children in front of us, so forcing today's children to encounter the difficulties of comprehension and apprehension which we learnt to solve for yesterday's children.

Mathematics today has a different structure from that which obtained 50 years ago in the traditional courses, so much so, that we require almost a complete refit to understand the situation. Modern algebra with its axioms and logical structure replaces the geometrical one which persisted from the Greeks although much of the terminology has a geometrical flavour. Algebra provides a rigorous basis reaching back to the theory of sets and of the natural numbers. Generalisations are easier algebraically than geometrically. Algebra brings precision, conciseness and mechanism to thought processes.

The algebra syllabuses must include theory of sets; considerations of number systems, groups, rings and fields; vector algebra; transformations; matrices; homomorphisms and isomorphisms.*

It would be out of place to give a detailed scheme based on this syllabus; instead let us consider briefly the ideas of groups which might permeate the course and deal in geometry with some examples of the other topics.

The theory of groups has probably more beautiful and astounding applications at a level beyond the maturity and sophistication of the school child. It is a simple example of a mathematical theory based on a few clear axioms. It is not an isolated topic but an integrating principle over the whole domain of mathematics.

We start with a composition law $(\cdot)$ which tells us how to combine two elements of a set S to produce a third element. From this law, or otherwise, we assert the following axioms

1. Closure: If $a, b \varepsilon S$ then $a \cdot b = c \Rightarrow c \varepsilon S$
2. Associative law: If $a, b, c \varepsilon S$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot b \cdot c$
3. Neutral Element law: If $a \varepsilon S$, $\exists\ \varphi \varepsilon S$ such that $a + \varphi = a$
4. Inverse Element law: If $a \varepsilon S$, $\exists\ a' \varepsilon S$ such that $a + a' = \varphi$.

A further possibility makes the above, which constitutes a group, into a commutative group:

5. If $a, b \varepsilon S$, $a \cdot b = b \cdot a$.

It is clear that if the operation is addition or multiplication, the natural numbers, 0, 1, 2 . . . do not form a group for there is no number a' such that $3 + a' = 0$. 0 is the Neutral element for addition, 1 the neutral element for multiplication. We invent in turn the rational numbers, the real numbers to complete the two group structures. Of course if we are concerned only with whole numbers, positive and negative integers will provide a group structure for addition. Some infant schools use this idea.

The use of numbers and their group properties plays a big part in selecting a number system and its algebra to serve as a model for a particular physical system. Thus weight uses the positive numbers without axiom 4 although the use of balloons could lead to an interesting discussion on this point. Length uses the real numbers and a full group structure. Considerations of different systems of measurements

* See Prof. Choquet's article in *Mathematics Teaching No. 14, Nov., 1960.*

would provide examples of groups and non-groups and so strengthen the concept. Often in modern science, a branch of mathematics is applied to it after a careful consideration as to whether its axioms apply in the physical situation, for if they do, then any arguments in the mathematical field will apply in the scientific context.

Another very simple example of a group is the differentiability of the addition of functions and this provides an example of an early use of homomorphic theory. A homomorphic mapping, $\varphi(x)$, may be defined by $\varphi(a \times b) = \varphi(a) \times \varphi(b)$ where $\times$ is the group operation. Its kernel $K = \{ x \mid x \varepsilon G \text{ and } \varphi(x) = \phi \}$ where G is the group and ϕ is its neutral element. An important theorem

$$\{x \mid \varphi(x) = \varphi(a)\} = a \times K$$

has many applications. For example, if we can solve $\varphi(x) = \phi$ to obtain the set $\{x \mid \varphi(x) = \phi\}$ then knowing one solution only of $\varphi(x) = a$, $x = b$ the complete set of solutions is $\{x \mid x = y + b, \varphi(y) = \phi\}$. We commonly use this theorem to solve differential equations, and systems of indeterminate equations, say n simultaneous equations in m variables with $m > n$.

We turn now to Geometry. It has been said that Geometry is on its way out in modern university mathematics. For a research mathematician it may be difficult to distinguish algebra from geometry but the vocabulary he uses is based on the synthetic geometry of the traditional courses. The schools have the very important task of keeping alive the spirit of synthetic geometry. But one must hasten to add that we should create a new basis readily understandable and apparent from physical observations, from the familiar rectilinear and curved objects with which school geometry is traditionally concerned.

In the earlier stages much geometry will appear to be familiar but it must be only as a set of neighbourhoods in the wider topology of mathematics. This is the new attitude; the goal is not the nine point circle although some students may well turn along its pleasant bye-road for a short diversion; instead we must look along the main high road toward a reappraisal of our physical approach in terms of the algebra whose main topics are listed above. We as teachers will start with a set of axioms towards which our students will work as a new foundation, a new starting point for adventures in mathematics which the schools at present do not enjoy.

My personal belief is that whatever the logical structure of geometry in the first fifty years of the next century, the structure probably best fitted to our present age will be based on a study of the algebra of directed line segments, one might almost say, fused with a study of symmetry, for from these ideas we achieve the A level starting point of more rigorous algebra. The axioms at this level will be

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathbf{a} + \mathbf{b} &=& \mathbf{c} & \lambda,\mu \ldots & \varepsilon & \text{Real numbers} \\
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &=& (\mathbf{a} + \mathbf{b}) + \mathbf{c} & \lambda(\mathbf{a}) &=& (\lambda\mathbf{a}) \\
\mathbf{a} + O &=& \mathbf{a} & \lambda(\mu\,\mathbf{a}) &=& (\lambda\mu)\,\mathbf{a} \\
\mathbf{a} + \mathbf{a}' &=& O & \lambda(\mathbf{a} + \mathbf{b}) &=& \lambda\,\mathbf{a} + \lambda\,\mathbf{b} \\
\mathbf{a} + \mathbf{b} &=& \mathbf{b} + \mathbf{a} & & &
\end{array}$$

and the inner product $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \lambda$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

These vectors have been familiar to students of mechanics for decades but that they form a basis for geometry with many simplifications of traditional difficulties has not been realised.

The above axioms are readily verifiable on a physical basis. $\mathbf{a} + \mathbf{b}$ may be achieved either by a parallelogram or a triangle, as in the case of the parallelogram or triangle of forces. The inner product $\mathbf{a} \cdot \mathbf{b}$ may be defined as ab cos θ although there are some other methods of arriving at its meaning. (Here is a research topic which calls for an appropriate teaching aid).

Pythagoras' theorem follows as a particular case:—

If $\mathbf{c} = \mathbf{a} + \mathbf{b}$ then $c^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$

giving the familiar cosine rule and the special case of Pythagoras when $\mathbf{a} \cdot \mathbf{b} = 0$.

Vectors and coordinate lattices provide the representation (x, y) for a vector $\mathbf{v}$ and $v^2 = x^2 + y^2$. If $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ it is easy to show that $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$ and the fact that $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ is a natural consequence of the result when $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. We have here a Stage B approach to vector algebra.

If $\mathbf{t}$ is a unit vector along a line, l, and $\mathbf{a}$ is the position vector of a point on it then $\mathbf{v} = \mathbf{a} + \lambda\mathbf{t}$ is the equation of the line l specifying all points which lie on it in terms of the parameter λ. A special case is provided by the coordinate equations

$$x = a_x + \lambda t_x,\ y = a_y + \lambda t_y.$$

If $\mathbf{a}$ and $\mathbf{b}$ are two points on l then we have the two forces

$$\mathbf{v} = \mathbf{a} + \lambda(\mathbf{a} - \mathbf{b}) \text{ and } \frac{y - a_y}{a_y - b_y} = \frac{x - a_x}{a_x - b_x}$$

Coordinate Geometry and vectors are alternative symbolisms for the same geometry but with the added advantage that generalisation to more dimensions is easy. For example a line in 3 dimensions is still $\mathbf{v} = \mathbf{a} + \lambda(\mathbf{a} - \mathbf{b})$ but the three coordinates which are the components of the vectors yield immediately

$$\frac{x - a_x}{a_x - b_x} = \frac{y - b_y}{a_y - b_y} = \frac{z - b_z}{a_z - b_z}$$

Transformations of geometrical figures by the isometries, similitudes, reflections, displacements and rotations give further results and provide applications of group theory and algebraic transformations. In many cases it will be more rewarding to consider the geometry of a simple set of points constituting a figure under various transformations than to consider general properties of figures.

This approach makes it simpler to isolate those properties which are independent of distance and angle. Affine geometry will thus be a direct application of vector algebra: the introduction of different metrics is made much easier if the description is algebraic. We may still send out people to study the sciences and their applications without detriment to their success but in addition and most important, the transition to University mathematics would be easier for our pupils. The goal of mathematics would lie on a continuation of the school high road.

The Association has a tremendous part to play in developing a new structure. Other societies may take on much of the purely mathematical development. Four tasks are before us:—

1. We must investigate with *our* children what problems are consistent with their immature sophistications and youthful pretensions. We must look for a set of fundamental mathematical properties which are discernible in the physical world and which maturity and depth of study will refine into a set of self-consistent axioms.

2. We must seek a perspective which has two directions: one from birth onwards, the natural growth of mental attributes and powers, the other from the mathematical maturity of our present age downwards through the school course.

3. We must seek for creative situations within the secondary framework which are not mathematically trivial.

4. We must marshall our available resources of teaching aids to investigate their possibilities in new fields of mathematics and we must seek for new learning aids and classroom practices which will lead to the efficient attainment of mathematical ends.

Our aim of founding progress on our pupil's reactions to the stimulus of the situation we create has already shown us that what is named "advanced" is a figment of the school teacher's own peculiar history of experience. Our objective in all our endeavours must be to develop . . .

qualities of mind and indeed of character which can face the challenge of strategic problems for which there is no stereotype; people of sound background who have not lost originality, initiative or perseverance in the course of training; people in whom facts have not ousted thoughts.

The Association is faced with an immense challenge to achieve positive results in the next few years.

Topics from the Mathematics of Sets and Relations

The problem of mathematics teaching, said **Professor Georges Papy,** was different from before. We taught mathematics no longer to volunteers but to conscripts. Yet if it were true that mathematics was taught to a minority then our task was impossible; but luckily it was not so. Mathematics had pervaded physics, chemistry and biology through to sociology, and had thus effected a change in the relations between itself and society. Two thousand years ago Euclid had written all his mathematics within the framework of perfect space; this was barren and had no relation to daily life. Now, however, mathematics was omnipresent, and pupils were attracted to it.

A new framework had sprung from non-Euclidean geometry, which led to many geometries instead of just "geometry", and in turn to many algebras. Prof. Papy used to think it foolish to combine arithmetic and geometry into one discipline, and he was right when geometry was the physics of space. Afterwards he understood that he was right not to understand. He did not answer as he thought; he answered as his teachers liked. We had probably lost many who were more honest.

There was a similarity in world syllabuses, but did a bird's-eye view present mathematics as a unified science? We had Euclidean geometry and arithmetic. We had added algebra, which was orientated arithmetic. Did trigonometry add orientation to geometry? In other words, could one attach a sign to an area? The answer depended on which professor asked the question, i.e., the answer depended on the authority. In Euclidean geometry we were forbidden to use the tools we knew existed in analytic and descriptive geometry. Science introduced new concepts which were not mere additions to previous ideas, because some of those must be discarded.

With good reasoning we could find good results in a particular sphere, but the same reasoning applied in a different sphere could have results which were incomprehensible to people at the time. In such a way began modern mathematics (in 1870) with the theory of sets, with which was included the theory of relations and functions. From this sprang the two branches, algebra and topology. Algebra was really calculation, by which from two elements one determined a third. Geometry, then, was calculation, in which two lines determined a point and two points a line.

It was usual to proceed from sets to topology, but one must start with topology in order to be able to understand sets. The blame for this reversed state of affairs was put on the universities by the schools and vice versa. This was irrational, "and so in Great Britain quite possible"! Topology introduced the idea of continuity, which could be explained in terms of actions producing results, and slightly changed actions producing slightly changed results. (An example of discontinuity was the rule: "if you obtain half-marks or more you will be promoted.") No school text book taught symmetry as a continuous function, but it was obviously so when a pupil produced continuously the reflection of a curve drawn by the teacher on the blackboard. A further example of discontinuity was the "carnival" function! In Belgian style, people marched in a circle, some carrying flags. When each person in turn indicated the next flag bearer in front of him this was easily seen to be a discontinuous function.

Prof. Papy had taught sets to children from eight to twenty-five years old, and it was more difficult with the twenty-five year olds! Undergraduates were conditioned by the bad habits of traditional mathematics; for example, they worried in Euclidean manner about whether a set was in a plane or in space. Children of eight or ten were not so conditioned, and most success transpired with some fifteen-year-olds so poor in mathematics that they were uninfluenced by previous courses!

One could begin, say, with ten-year-olds, who already had the idea of a set. It was unnecessary to tell them, as did a French text-book, "we shall call a set any collection of objects." They knew sets in everyday life, a set of classes in a school or a flock of sheep. Did a glass belong to a flock of sheep? No! Why not? It was not a sheep. If to every object we could reply in the affirmative to the same question then we had a set.

Sets need not consist of objects which were naturally together. Could one construct more amusing sets? He had dispensed with suggestions of a set of clowns or toys, and had arrived at a carafe, a watch and Prof. Papy! Could any stranger entering guess the elements of the set? No, for it had been decided by the class (something new in mathematics!). How could one indicate this set to a foreigner? Put all the elements in a large bag. (This was dangerous; the A.T.A.M. were implored not to produce large bags!) Eventually a pupil (with permission!) had drawn a chalk circle on the floor round the objects.

Since abstract things must never be explained with abstract words, the set had been drawn on the blackboard, not pictures of objects, but only the circle. Pupils had been able to indicate the positions of various objects in the room on the blackboard, according to whether or not they were in the defined set. No points were drawn until it was necessary in order to avoid the impression that a set was always finite. This was not news: we did not name all the points on a line. An example of

an empty set had been given by the class—the set of fish in a place where there were obviously no fish!

If a flock of sheep were in the room, then so were their tails. But did the tails belong to the set of sheep? After this shock the class had been able to appreciate that the minute hand of the watch was not in their original set, and on the blackboard diagram would be placed outside the circle. Similarly, Prof. Papy's nose did not belong to the set of instructors, since it was not an instructor. Further examples of sets given by the children had shown that care was necessary in defining one's elements. When the set of blonde girls had been suggested, two girls, one blonde and one black, had differed as to whom they considered to be blonde.

Consider the set of flowers in a shop, then the marguerites, then the white flowers. The diagram for these introduced the need for notation. Two intersecting sets, called A and B were drawn on the blackboard. Notation was usually introduced in the following order:

(i) $A \cap B$, the set of elements in both A and B
(ii) $A - B$, the set of elements in A but not B
and $B - A$, the set of elements in B but not A
(iii) $A \cup B$, the set of elements in either A or B or both.

Now from a complicated diagram of sets children could be asked to express each in various ways. Later, ϕ could be introduced for the empty set, and $B \subset A$ for "B is included in A." One child asked to draw $B \subset A$ had drawn two intersecting sets, but explained that $B - A$ was ϕ. It was easy to see that A and B were *disjoint* if $A \cap B = \phi$, and to formulate such laws as

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

using diagrams and coloured chalks. Logic was unnecessary. The *diagram* was our logic.

The only question for which there was time was "What use is all this?" By way of explanation, Prof. Papy explained that pupils accustomed to ordinary mathematics never gave *mathematical* examples of sets. When asked specially to do so, they offered the set of theorems or set of definitions, but never gave the line as a set of points. One could partition the plane into lines as subsets, so that each point of the plane belonged to one line. Prof. Papy himself had been worried by the statement "a line divides a plane into two regions." What about the points on the line? His teacher had suggested he cut along the line. But two intersecting cuts along lines which intersected at a point then proved an interesting problem. Incidentally, young children, when being asked how many points were on a line, considered that between any two points there was room for another point—or even half a point!

Demonstration Lesson

Professor Papy began in silence by putting an array of dots on the blackboard, and since there was one empty desk in the class the first-year grammar school girls realized that they and not the desks were being represented. They eventually became more used to Prof. Papy's accent as well as the strangeness of the situation, and understood after some initial difficulty that they were being asked to indicate anyone in the class whose Christian name began with the same letter as their own surname,

and that this relationship was to be represented by arrows on the diagram. The blackboard soon became covered with arrows, and it was interesting to see where the successive arrows were placed as Prof. Papy begged them not to "destroy the graph." Unfortunately the teacher with the class, who was checking on their accuracy, anticipated one girl who had to indicate herself !

Prof. Papy drew another diagram, complete with arrows, which he said represented another class, and the girls could see that certain information could be derived from it.

Next, a haphazard array of dots, he said, represented boys and girls playing in a field. Could they distinguish boys and girls? Obviously not, they decided. This time, a red arrow would represent the relation "he is my brother." One arrow was inserted, and it was insisted that this was the only arrow possible. What could they tell now? Obviously the arrow pointed to a boy. It was some time before they realized that it must come from a girl, since there was no arrow in the reverse direction. Then an arrow was placed from the boy to a third child. Was this the complete picture? No, since the third child would recognise the boy as a brother, and would be recognised as a brother by their sister. The diagram became more and more complicated, and green arrows were added to indicate the relation "she is my sister."

Teaching Modern Mathematics

In this lecture **Professor Papy** had promised to comment upon his lesson, but first he gave an example of an application of set theory. We had met the common calculation given to pupils

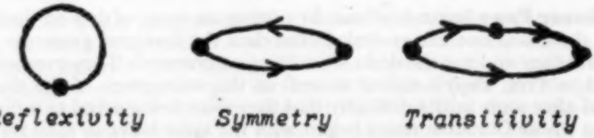
$$(b_1 + r_1)(b_2 + r_2)(b_3 + r_3) = ?$$

in which a product of sums is to be replaced by a sum of products. This had no application, and so no motivation. But replaced by

$$(b_1 \cup r_1) \cap (b_2 \cup r_2) \cap (b_3 \cup r_3) = ?$$

it could be applied to the theorem on internal and external bisectors of angles. The other concurrence theorems could be proved in a similar way.

It was impossible to develop advanced mathematics without the concept of function. (Even Euclid used relations and functions, without actually putting them into words). Hence the idea should be introduced at the first moment of appeal, instead of when it was impossible to do without it, i.e., at a point of difficulty. A *relation* was a set of pairs of points. In the lesson this abstract idea had been presented in a concrete way, and the two connected by a blackboard diagram. There were intersections and inclusions of relations as there were of sets. Each arrow had a reverse arrow; "l'autre flèche," his Belgian pupils had called it, but a consistent terminology was needed. Ideas of reflexivity, symmetry and transitivity were introduced as in Fig. 2.



Children could understand functions as soon as they understood phrases such as "has a father." Such relations avoided the notion of universal symmetry. One must always give negative examples in order to clarify generally accepted ideas; for instance examples of non-symmetry could elucidate the idea of symmetry.

Products of functions could be introduced by graphs similar to those used in the lesson, by considering, say, a yellow arrow followed by a blue arrow replaceable by a red arrow. Such a concrete example would then lead to abstract reasoning. Difficulty was experienced by dull children who not only forgot questions but were unable to view the situation as a whole; wrong solutions from them did not imply that *all* functions were misunderstood. There was not always a unique solution, and there was not necessarily a *verbal* answer. Therefore clarity of diagram was important. Good answers were sometimes badly presented.

A "bi-function" had been an invention of pupils, and had led to the notion of isomorphism. Defining a primary number as a power of a prime, consider the sets of primary divisors of 48 and 36

$$P(48) = \{2^4, 2^3, 2^2, 2, 3, 1\}$$
$$P(36) = \{2^2, 2, 3^2, 3, 1\}$$

Then

$$P(48) \cap P(36) = \{2^2, 2, 3, 1\} = P(12) = P(48 \cap 36)$$
$$P(48) \cup P(36) = P(48 \cup 36)$$

were the H.C.F. and L.C.M. respectively. Here we had an example of distributivity.

Prof. Papy swiftly passed on to Stone's theorem on lattices and Pasch's theorem about the preservation of order in parallel projection, saying as he did so that one must go slowly for pupils—and for teachers!—for in presenting mathematics we knew, the others had to think.

Consider the real line between 0 and 1, on which was marked the point a. Bisect the interval (0, 1), and write 0·0 or 0·1 according as a is to the right or left of the bisection. Assuming it is to the right, bisect the interval $(\frac{1}{2}, 1)$ and write 0·11 or 0·10 according as a belongs to the right or left set. Continuing in this way, and assuming only a knowledge of sets and the possibility of bisection, one obtained a binary expression for the distance a. Clearly decimal notation could follow.

Prof. Papy ended with an alternative to Mr. Hope's vector proof of the extension to Pythagoras' Theorem, in which the cosine was defined in terms of the scalar product instead of vice versa.

There was time only for the one question "How can we find all this out?" The simple answer was "Study!"

What Next?

"Sometimes we would like life to be a succession of conferences," said **Mr. Wheeler.** We met others doing the same job and experiencing the same difficulties in spite of their varied backgrounds, and this gave us a feeling of fellowship. However, what were we going to do when we went back to school? The challenge of the situation was now more real than it had been on the first evening. We had liked Prof. Papy's lesson, but what came next, where did it fit in, and how could it be developed? It stood in its own right as something which educated children, but we had to learn,

teach and carry out our own researches in order to see its place in the rest of mathematics. It was not necessarily dangerous to teach modern mathematics in the wrong way—we could do no worse than we were already doing!

The discussion that followed expressed two things—a thirst for information and dissatisfaction with the conservatism of existing syllabuses. Books were recommended, but immediate action was taken in the form of study groups, to be organised by leaders all over the country who volunteered in response to Mr. Wheeler's request. The A.T.A.M. could also play its part in exerting more pressure on official bodies, in forming more branches to exert this pressure, and in demanding courses from the appropriate authorities.

The conference ended in this spirit of enthusiasm. It had been numerically successful, said Mr. Wheeler, and it was not for us to say anything else. A string of thanks was unnecessary, and any good effect was its own thanks. An exception could perhaps be made in the case of Prof. Papy, for his "inspiration to our affection and admiration."

BOOKS FOR STUDY IN MODERN ALGEBRA

The following list is intended to give help to anyone not very familiar with books in this field. (The prices quoted are in some cases approximate). It has to be remembered that the majority of books on modern algebra are written for undergraduate students. The best books for our particular purpose have yet to be written.

(A) Introductory Books:

Prelude to Mathematics (chapters 7 to 14). W. W. Sawyer (Pelican, 1955: 2s. 6d.)

Modern Mathematics with Numbers in Colour (chapter 1) C. Gattegno (Cuisenaire, 1959: 5s.).

Sets, Sentences and Operations. Booklet in series "Exploring Mathematics on your own." (Webster Publishing Co., St. Louis, Missouri: 5s.).

(B) Study Books:

Premiers Eléments de Mathématique Moderne. G. Papy (special order from Ecole Normale de l'Etat, 72 Rue Berkendael, Bruxelles, Belgium: £1).

Introduction to Modern Algebra. J. L. Kelley. (van Nostrand, 1960: 21s.).

A Teacher's Guide and Student Manual to this course are available from the same publishers.

A Concrete Approach to Modern Algebra. W. W. Sawyer (Freeman, 1959: 7s. 6d.)·

Modern Abstract Algebra. R. V. Andree (Constable, 1958: 42s.).

(C) Reference Books:

A Survey of Modern Algebra (Revised Edition) Birkhoff and MacLane (Macmillan 1953: 45s. 6d.).

Finite Mathematical Structures. Kemeny, *et al* (Prentice Hall, 1959: 50s.).

The Skeleton Key of Mathematics. D. E. Littlewood (Hutchinson, 1949: 10s 6d.).

AFTER THE CONFERENCE

Many of the members of the Easter Conference felt the need for further study of the elements of modern mathematics so that they would understand better the significance of the work being done by Professor Papy and perhaps be in a position to take part in similar experiments in this country. At the closing session of the Conference several members volunteered to act as local secretaries for the formation of small study groups in their areas. The aim of the groups will be to provide opportunities for members to learn together in this subject. If you are interested in this project and live near enough to one of the people named below for attendance at regular meetings to be possible, will you write to him straight away? Details of these groups and any others that come into being will be published in future issues.

If you live in an area too far from any of the secretaries, can I suggest that you try to create a demand for a group and get in touch with your local H.M.I., or Education Authority or University Institute of Education and press them to provide a course for you? I will gladly write in support of a demand by a group for such a course if this will help in any way. I have already sent letters summarising the feeling of the Conference and stressing the urgent need for study courses in modern mathematics to A. P. Rollett, Esq., H.M.I., to the Secretaries of the Association of Education Officers and the Association of Education Committees, and to the Directors of all the University Institutes of Education in England and Wales. It is up to members to make this general demand more specific so that some action will result.

Group Secretaries

A. W. Bell, Nottingham Training College, Clifton, Nottingham.
Miss J. Blandino, 9 Barnhill Road, Wembley Park, Middlesex.
W. M. Brookes, 45 The Parkway, Bassett, Southampton.
A. R. Colyer, 199 Henley Road, Caversham, Reading, Berks.
W. G. Cooper, Briar Bank, Sandy Lane, Woodbridge, Suffolk.
J. G. Evans, Wolverhampton Day Training College, Wolverhampton, Staffs. (from September).
D. S. Fielker, 90 Burnt Ash Hill, Lee, London S.W.12.
F. Gorner, Department of Education, The University, Manchester 13.
Miss S. Jones, Maghull Grammar School, Maghull, Liverpool.
J. E. Kirby, Whitegate House, Podmore, Scarning, Dereham, Norfolk.
C. Steele, Hillside, Old Lodge Lane, Kenley, Surrey.

Before leaving for Canada, Trevor Fletcher independently announced that he would hold a group of this kind starting in the autumn. His address is 148 Lennard Road, Beckenham, Kent.

The establishment of even a dozen small groups working at this task could prove of enormous value to the profession and eventually to the children. This project is not a disinterested pursuit of knowledge for its own sake (although there will be much individual benefit on this account) but a small attempt to help us all to teach better. I hope with all my heart that this scheme prospers and grows in extent and influence.

D. H. WHEELER.

A.T.A.M. RESEARCH GROUP

A research group of 14 members was formed at the Easter Conference. In a first investigation, three of Piaget's and Inhelder's experiments will be repeated with a wide age-range of children.

EIGHTH ANNUAL GENERAL MEETING

At the meeting on April 13th, 1961, at Berridge House, Hampstead, N.W.6., the Secretary reported on the year's work, mentioning the net increase of 186 members and the increase in size and printing of "Mathematics Teaching". The Treasurer presented the accounts and showed that the deficit at the beginning of the year had been converted into a surplus by the end.

The Secretary read a statement prepared by the Committee recommending an increase in the annual subscription to the Association. The meeting approved the raising of the subscription to £1 a year to take effect from October 1st, 1961, and added a recommendation that the Committee should consider instituting a reduced subscription rate for students.

A list of officers and committee members for 1961-62 appears on the back page of this issue.

D.H.W.

ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS

INCOME AND EXPENDITURE ACCOUNT FOR YEAR ENDING DECEMBER 31ST, 1960

EXPENDITURE	£ s. d.	£ s. d.
Printing & Distribution		
Bulletin No. 12	180 18 3	
" No. 13	187 17 10	
" No. 14	201 1 9	
	569 17 10	
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		574 11 10
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BALANCE SHEET

As at 31st December, 1960

LIABILITIES	£	s.	d.	£	s.	d.	* ASSETS	£	s.	d.
Subscriptions prepaid..................				89	15	0	Cash at Bank..........................	127	3	7
Printing due and unpaid. Bulletin No. 14.				4	14	0	Adverts due and unpaid...............	39	9	0
Surplus..................	143	11	4							
Less Deficit 31/12/59.......	71	7	9							
				72	3	7				
				£166	12	7		£166	12	7

I certify that in my opinion the above statements of account are properly drawn up so as to exhibit a correct and true view of the Association's affairs according to the best of any information and the explanations given to me and as shown by the books of the Association.

* This balance takes no account of sums of £12 3s. 4d. and £9 6s. 6d. held on imprest by the Hon. Secretary and Hon. Editor respectively.

(Signed) K. S. COLLINS.
26th March, 1961.

LETTERS TO THE EDITOR

Sir,

While there is little doubt that most of the people who attended the Berridge House Conference at Easter enjoyed it and found it stimulating, many came away unconvinced of the case for introducing modern mathematics in schools. From my own experience I can guarantee that those who respond to Professor Papy's advice ('Study!') will find themselves rewarded by the interest and beauty of the modern approach, but the problem facing many busy teachers is how to make the best use of their limited time in the interests of their pupils.

Since the Conference I have re-read Professor Choquet's article in *Mathematics Teaching*, No. 14. It corresponds largely with the material presented by Professor Papy at Berridge House. As I understand him, Professor Choquet objects to the mathematics customarily taught because it does not train the pupil's selective and creative powers. He dismisses as aims for teaching the presentation of a series of 'things that every technician, engineer and scientist must know' and also 'the aim . . . to train our pupils to think correctly and logically' on the grounds that 'machines are gradually taking over the tasks of memory and calculation and replacing human labour'. So far I follow him, and, in fact, the trend of enlightened teaching, with its lessening emphasis on bookwork, the lightening of the figure-load and the admission of aids to calculation has lately derived additional support from the development of memory/calculating machines. The meaning of the rest of Professor Choquet's article, except the strictly mathematical illustrations, is hidden from me. I can hardly believe we are intended to work out a scheme for beginners in the light of these illustrations. However, the summary of Professor Papy's scheme of lessons which Mr. Harris has promised to publish may perhaps resolve this difficulty.

Frankly, Sir, quite a number of members of the Conference were bewildered by what acceptance of Professor Papy's thesis would mean. Many of them felt that they ought to undertake the study he recommended, but they would like to have clearer evidence that it would be more valuable for their pupils than what they would have to give up. In particular, it would help to know how relevant the proposed work is to children of lower intelligence.

That conventional school mathematics contains no ideas less than 300 years old is hardly *in itself* enough reason for introducing more recent ideas. What is needed now is a clear statement, *written* in English (not translated), by a teacher of mathematics or science, or an industralist, explaining the relevance of modern mathematics and its relationship with the conventional work. Whatever Professor Choquet may have written, the conventional work is clearly needed at present for science and technology as well as for examination purposes.

What is the evidence that a pupil who has studied some modern mathematics can cope later with conventional work more rapidly than one who has not?

For my part, though I have spent some time (not enough!) on study, I cannot yet conscientiously see anything I can leave out of my present college syllabus to make time for modern mathematics.

Finally, Sir, may I express the hope that the activities of the A.T.A.M. and the policy of *Mathematics Teaching* will continue broadly based, as they always have been? Rank and file members depend on the Association and the Bulletin for help and it would be a pity to give undue emphasis to any one aspect of the subject, however influential its protagonists may be.

Yours, etc.

C. T. STROUD.

Newland Park College,
Bucks.

(On Mr. Stroud's last paragraph we would assure him and our other readers that Editorial policy is always to present a balance between the different types of articles appearing in each issue. These must include articles dealing with different aspects of teaching the subject from primary to advanced level, as well as articles which encourage the reader to keep abreast of modern developments in his subject. —*EDITOR*).

Sir,

Thank you for letting me see Mr. Stroud's letter to you.

Those who attended the Easter Conference will be in a position to decide for themselves how far Mr. Stroud has succeeded in conveying the challenge that was presented there, but I think that one point ought to be made clear to our other readers. Professor Papy not only believes that some of the concepts of modern mathematics should be worked into a scheme for beginners, but he gave us some pretty clear illustrations of ways in which this could be done. The territory is not quite so uncharted as Mr. Stroud's letter implies.

I have every sympathy with any teachers who came away from the Conference undecided about the practical value of Professor Papy's suggestions. These are too new and too strange to all of us for us to be quite certain yet of their relevance to our teaching. What does make me unhappy is to think that we ought to wait for evidence of this relevance *before* we study the suggestions or test them in our classrooms. If

we accept Mr. Stroud's view in general terms we are implicitly refusing responsibility for future developments in the teaching of our subject.

Mr. Stroud may, as he hints, be speaking for a substantial number of your readers. He is not speaking for all.

Yours, etc.

DAVID WHEELER.

Lecturer in Education,
University of Leicester.

SALE OF LITERATURE

Back numbers of *Mathematics Teaching* and copies of *Mathematics Teaching Pamphlets* may be obtained from the Literature Secretary:

Mr. J. Tyrer, 4 Downs Road, Maidstone, Kent.

Nos. 1 to 4 of *Mathematics Teaching* are out of print, but other issues are available in limited quantities. No. 5 costs 3s., Nos. 6 to 8 cost 2s. 6d. each., and the rest are priced at 3s. 6d.

Mathematics Teaching Pamphlets include the following titles to date:

No. 1 Film and Filmstrip list.
No. 2 Working Model Mathematical Wall Charts.
No. 3 Looking at Rhombuses.
No. 4 Looking at a Regular Hexagon.
No. 5 Maths with Colour Rods—Beginnings.
No. 6 Maths with Colour Rods—Beginnings of Written Work.
No. 7 Maths with Colour Rods—Addition and Subtraction.
No. 8 Paper Folding (a reprint of the article appearing in *Mathematics Teaching*, Issue No. 14).

Nos. 1, 2 and 8 cost 1s. each, the rest 6d. each (post 2d. extra on each order).

COMPUTER DEMONSTRATION

Considerable interest was created by the demonstration at the Easter Conference of a Computer made by the boys of Trinity School of John Whitgift, North End, Croydon. Mr. M. D. Meredith, who was in charge of the project, has undertaken to write a fully illustrated article about this computer for publication in one of our future issues. However, as there will be inevitably some delay in the publication of this article, Mr. Meredith has offered to send details in advance of this to anyone requiring the information at once. When writing to Mr. Meredith at the school, please enclose a stamped and addressed foolscap envelope.

APPARATUS REVIEW

Angle Properties of the Circle

Wood and elastic model by Mathematical Pie Ltd. 25s.

This model consists of a plywood circle inset in a one foot square plywood board so that the circle is flush-fitting with the surround. By means of drawing pins it is possible to fix points on the circumference, and these may be either movable (by putting the pin in the movable circle) or fixed (by putting the pin in the surround). Tangents, chords and angles in the circle are represented by round elastic looped round the drawing pins, and the model allows the dynamic aspects of circle properties to be demonstrated. The flush fitting of the circle and its surround are particularly advantageous in the smooth working of the model, and another special feature is that the split pin representing the centre may be easily removed to allow free movement of the elastic across the central position.

The model is quite effective in use and should be of value particularly to those people who do not find it easy to make their own models for class demonstration. The one disadvantage we found in use was that the model is finished in matt paint of pastel shades; this looks very good when the model is new, but soon shows grubby finger-marks which are not easy to remove. We wonder if the model would not be better finished in gloss paint.

C.B.

Constructional Models for Simple Solids

Prepared by A. Ivell. University of London Press Ltd. Price 3s. (+ tax 7½d.)

This apparatus consists of an envelope containing four large sheets of stiff white cardboard precut with the nets of various solid models. The models catered for are, one 2 in. cube, nine 1 inch cubes, one tetrahedron, one right rectangular prism, one hexagonal prism, one octahedron, one cylinder and one right circular cone. The omission of a square pyramid and a triangular prism is difficult to understand.

The models are easy to construct, having convenient tabs for the addition of glue or other sticking solutions; a contact adhesive was found to be the best. the cylinder and cone need careful manipulation, a good curved surface being obtained by the use of a thick rod along the inside.

These solids are useful in all types of school and very young children show great interest in making, comparing and exploring them. At a later stage individual work can be done with them, and the writer found it useful to have models in various stages of construction, as in television demonstrations.

Median lines and perpendiculars can be drawn on the models to illustrate various properties. The models are of convenient dimensions which facilitates practical work in measurement and subsequent calculation of areas, volumes, etc. Fundamental concepts of volume can be demonstrated and proportional relationships between the sides and volumes of two different sized cubes can be brought out. More advanced notions are possible. Models can be left with one final flap unsealed

and the solid filled with dry sand or similar material which can then be poured into other models to compare and elicit such relationships as that between the volume of a cone and its surrounding cylinder.

The models are extremely useful and versatile at all stages.

S.W.B.

The Aristo Geo-liner 1551

Dennert & Pape, Hamburg. British Distributors: Technical Sales, 32 & 32A Lupus Street, London, S.W.1. 2s. 9d. each

In appearance this instrument resembles a plastic 45° set-square of hypotenuse about 6 inches. However, the Geo-liner incorporates a protractor marking and the degree markings are carried out to the two shorter sides, while the hypotenuse is graduated in inches and tenths starting from the centre and reading outwards each way. A number of lines parallel to the longest side are also marked on the instrument at quarter-inch intervals. The various uses of the instrument include use as a set-square or protractor, and also for drawing parallel lines and symmetrical figures. The big advantage of the instrument over the more conventional types arises from the fact that the graduated scale is marked from the centre of the hypotenuse. This enables distances to be scaled off at the same time as angles or perpendiculars are drawn, and with the zero mark at the centre there are none of the usual disadvantages when corners of the instrument become damaged. The Geo-liner is well-conceived, the scales are engraved and clearly marked, and it proved durable in use.

C.B.

MATHEMATICAL FILMS

The following three films, all of which are available on free loan, make a well-balanced programme about computers. "Introduction to the Pegasus Computer" gives a good idea of what a computer is like and how it is operated. The film is in colour, runs for about ten minutes and is obtainable from Ferranti Ltd., The London Computer Centre, 68/71 Newman Street, London, W.1.

"Leo the Automatic Office", (black and white, 15 minutes) which shows another installation at work on commercial computations such as payroll and stock control for Lyons teashops, can be borrowed from Leo Computers Ltd., Hartree House, 151a–159a Queensway, London, W.2.

The third film, "The Information Machine, or Creative Man and Data Processor" is a cartoon but a cartoon with a serious purpose. International Business Machines commissioned Charles Eames to produce this eight minute colour film for the Brussels Exhibition in 1958. With humour it shows the historical development of a situation in which computers extend man's power over his environment, and stresses their use in control, design and simulation. This can be obtained from the Education Department, IBM United Kingdom Ltd., 101 Wigmore Street, London, W.1.

Whilst all of these films were made for sales promotion they are free of any objectionable advertising, and are well suited for teaching use.

T.J.F.

RIGHT-ANGLED TRIANGLES AND SQUARE ROOTS

An Approach for the Secondary Modern School

RUSSELL PARKER

I have been asked to write something about my experience as a specialist teacher of Mathematics in a Secondary Modern School, and I would like to begin by saying that I think the current series in "Mathematics Teaching" on Secondary Modern School Mathematics is most useful and stimulating.

It is, however, somewhat depressing to read Mr. Walters' comments on the low mathematical ability of such a large number of Training College entrants ("M.T.", No. 10). One remark which he makes is perhaps particularly relevant to this situation: ". . . building new ideas . . . on the framework which the child already accepts and understands in such a way that he will accept the new as reasonable and not as a piece of magic which he must simply learn and memorise". One fears that in many cases the Training College student who fears and dislikes Mathematics has been introduced to the subject in his school days as a subject in which you "Do this" and then "Do that" and at the end, by some mysterious process which is beyond mortal ken, you get the "Right Answer".

It is my own strong opinion, for what it is worth, that a real effort should be made to *explain* processes—or, ideally, to get the child to understand the process for himself—and that the temptation to take the easy way and get the child to, so to speak, jump through mental hoops and perform tricks, of the meaning of which he is entirely ignorant, should be resisted.

Of course, in the Secondary Modern School we are dealing with children who are not so quick mentally as their opposite numbers in the Grammar School. This fact must be taken into account, or progress will be nil. Simple words, simple analogies and an encouraging manner must be used. And a good thing, too. I could write quite a lot on what I think of the technical terms used in mathematics—nearly always of Latin or Greek origin, which are enough to put anyone off. Numerator, denominator, hypotenuse, perpendicular, characteristic, mantissa, abscissa, ordinate, to mention just a few of the elementary terms in use, could surely be scrapped in favour of less formidable words, which might give a clue to their real meaning, and which do not disguise the fact that the ideas behind all of them are *extremely simple.*

In the Secondary Modern School we are not only dealing with the *less able* child, as regards his Grammar School counterpart, but we are also dealing with the *average* child, when considering the child population of the country as a whole. This is rather an important consideration. It would be a great pity if 75 per cent. of the country's children should be condemned to nothing but the drudgery of computation, learning "tables" and the manipulation of our somewhat stupid system of coins, weights and measures. It is encouraging, in this context, to hear and read so much nowadays of the need to present to children the *ideas* of mathematics, and so to stimulate their interest and to afford them pleasure in a fascinating subject.

One of the early facts encountered in plane geometry is that the square on the longest side of a right-angled triangle is equal in area to the sum of the squares on the other two sides. This fact traditionally enjoys the name of "Pythagoras' Theorem", although practical applications of it were known long before Pythagoras' time—

in Egypt, for example, where the 3, 4, 5 triangle was used in the erection of the pyramids. This last historical fact is of interest to children, but I would suggest that the traditional name does more harm than good: it is an additional "tongue-twister" which has little relevance.

The right-angled triangle can be used for the solution of a number of practical problems, and in the initial stages careful choice of measurements will ensure that all sides of the triangle have lengths measured in whole numbers of units. Inevitably, however, and rightly, the question of irrational quantities will arise. How, for instance, can we determine the value of the square root of 2?

Well, there are of course a number of ways of tackling the problem. We can use logs. (if the children are acquainted with logarithms). There is the standard arithmetical method, often in the past taught in Grammar schools as a rule of thumb operation, to be memorized, and as quickly forgotten after leaving school.

I would suggest a good approach is the *iterative method*. Select a good, simple approximation to the required square root.

Number	1	2	4
Square root	1	?	2

From the above it is obvious that $\sqrt{2}$ is greater than 1 and less than 2. Considering the numbers in the top row, it is seen that 2 is nearer to 1 than it is to 4. We may therefore assume that $\sqrt{2}$ is nearer to 1 than to 2, i.e., it is less than 1·5. Let us take 1·4 as an approximation.

Now there are three possibilities:

(1) 1·4 is exactly equal to $\sqrt{2}$.
(2) 1·4 is less than $\sqrt{2}$.
(3) 1·4 is greater than $\sqrt{2}$.

If, therefore, we divide 1·4 into 2, the respective answers will be:—

(1) 1·4.
(2) A number greater than 1·4.
or (3) A number less than 1·4.

In case (1), we need go no further, for we shall have obtained our answer. In cases (2) and (3), the true value of $\sqrt{2}$ will lie between our first approximation and the answer to our division. In either case the arithmetic mean will give us a better answer. The process can be repeated, if necessary.

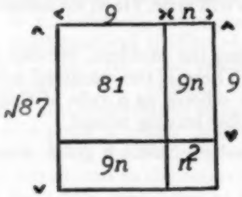
In the example chosen, we have $2 \div 1{\cdot}4 = 2 \div 7/5 = 2 \times 5/7 = 10/7 = 1{\cdot}428 \ldots$

Taking the mean, we have: $\frac{1}{2}(1{\cdot}4 + 1{\cdot}428 \ldots) = 1{\cdot}414$ which is equal to $\sqrt{2}$, correct to three places of decimals.

Memory—or, perhaps, lack of memory—of the traditional method for the extraction of the square root might well deter a teacher from broaching the subject with a Secondary Modern class. But the above method is so simple that it is a pity children should not be enabled to use it. Of course, at a later stage, they will use square root tables, but, again, it would be a pity to use these tables as a gift from some magician, when they can be shown that they too, can find square roots. This is how self-respect and self-confidence are bred.

The method described, incidentally, is a very ancient one, and was known to Hero, of Alexandria (c. 100 B.C.). The iterative method, apart from its use in finding square roots, is a valuable mathematical concept, later used with great effect by Newton. Here we have an *important* mathematical idea, and a *simple* one.

A method for obtaining the approximate value of a square root was given in "Mathematical Pie", No. 10, October 1953. Here we consider the number whose square root is required as an *area* of the stated number of *square units*.



To find root 87.

Let the large square represent 87 square units. The large internal square represents 81 square units. Let the upper side of the large square be $9 + n$ units. Then each rectangle has an area of $9n$ square units, and the small square has an area of n^2 square units.

From this diagram it is clear that $\sqrt{87} = 9 + n$.

Also $18n + n^2 = 87 - 81 = 6$.

n must be less than 1 and it is clear that it is small. Therefore n^2 will be smaller still. For our approximation we will therefore ignore n^2.

We thus have $18n \simeq 6$

$n \simeq 1/3$

Therefore $\sqrt{87} \simeq 9{\cdot}33\ \ldots$

and, from the tables, $\sqrt{87} = 9{\cdot}33$ to two places of decimals.

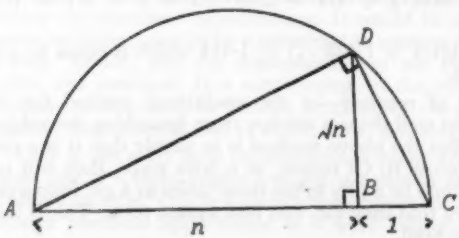
The result may be generalized to:

$$\sqrt{N^2 + n} \simeq N + \frac{n}{2N}$$

In a case such as $\sqrt{97}$ it would be advantageous to put

$$\sqrt{97} = \sqrt{10^2 - 3} \simeq 10 - \frac{3}{20}$$

There are other geometrical approaches to this problem which ought not to be overlooked. These involve measurement, which—like all measurements—can only be approximate. Use of compasses and dividers is suggested for greater accuracy. The first of these methods requires the knowledge that the angle in a semi-circle is a right-angle and also the relationship of the sides of a right-angled triangle.



In the diagram the distance AB represents n units, where n is the number of which we desire the square root. The distance BC represents *one* unit. The distance BD represents $\sqrt{n}$. The proof is easily demonstrated, and gives practice in geometrical reasoning and algebraic manipulation.

The second method is by direct application of the relationship of the sides of a right-angled triangle. For example, since $13 = 4 + 9 = 2^2 + 3^2$, then 2, 3 and $\sqrt{13}$ are the sides of a right-angled triangle. Thus, the triangle may be drawn with the sides including the right angle 2 and 3 units long and measurement of the hypotenuse will give a length $\sqrt{13}$ units. In the case of some surds there is more than one alternative triangle which can be used. Below is a list of sides for obtaining $\sqrt{2}$ to $\sqrt{12}$.

1·5	0·5	$\sqrt{2}$
2	1	$\sqrt{3}$
2·5	1·5	$\sqrt{4}$
3	2	$\sqrt{5}$
3·5	2·5	$\sqrt{6}$
4	3	$\sqrt{7}$
4·5	3·5	$\sqrt{8}$
5	4	$\sqrt{9}$
5·5	4·5	$\sqrt{10}$
6	5	$\sqrt{11}$
6·5	5·5	$\sqrt{12}$
.	.	.
.	.	.
.	.	.
$\frac{1}{2}(n + 1)$	$\frac{1}{2}(n - 1)$	$\sqrt{n}$

$(n = 2, 3, 4, \ldots\ )$

A teacher is likely to be asked by an intelligent pupil how many right-angled triangles there are which have sides measuring each a whole number of units. The answer is, of course, an infinite number, and the next question is: How can one find them? If m and n are whole numbers, and m is greater than n, then the lengths of the sides will be:

$$m^2 + n^2 \text{ (the longest side)}$$
$$m^2 - n^2$$
$$\text{and } 2mn.$$

This general integral solution to the equation $a^2 = b^2 + c^2$ was known to Diophantus (4th cent. A.D.). Surely, this is the place to bring to the attention of the class one of the most famous of unsolved mathematical problems, Fermat's Last Theorem, which has baffled mathematicians for three hundred years. Fermat (1601-65) stated that $a^3 = b^3 + c^3$ has no integral solutions, and similarly for higher powers, but no one has been able to prove this apparently simple proposition. In 1908 the late Professor Paul Wolfskehl, of Germany, left 100,000 marks to be awarded to the first person giving a complete proof of Fermat's Last Theorem. The inflation after the first World War, however, reduced this prize to practically zero!

Before leaving the right-angled triangle, it may be mentioned that the relationship between the sides applies not only to the squares drawn on them, but also to equilateral triangles, regular hexagons, semi-circles and other shapes (e.g. the semi-circle drawn on the longest side of a right-angled triangle is equal to the sum of the semi-circles drawn on the other two sides). The proofs of these are straightforward and give useful practice both in geometry and algebra.

BOOK REVIEWS

Ordinary Level Mathematics

L. H. Clarke, Heinemann, Second Edition 1959. 387 pages. 10s. 6d.
With Answers 12s. 6d.

This book follows the traditional method of teaching G.C.E. mathematics, but this does not detract from the fact that the book is very well compiled and contains a wealth of material for both the teacher and the revising pupil.

The presenting of percentage solely as a ratio, the use of the practice method in preference to the use of the formula and logarithms in the solving of compound interest problems, and the assessing of the characteristic in logarithms according to the "standard form", are three very questionable features in the book's many excellent contents. The geometry and calculus sections are very well presented.

Throughout the book information is formally stated with little if any explanation. Whilst this may be useful where a teacher can fill in the obvious gaps, it in no way helps the unaided pupil using the book to further his mathematical knowledge and understanding. As with so many books of this formal kind, it lends itself to rote learning without any real understanding.

Every section is exceptionally well indexed and is prefaced by useful notes and formulae and there is a valuable set of problems at the end of each section. In the solving of problems the reviewer highly commends the author for his insistence upon rough estimates prior to the calculation, and a check being made on answers obtained.

Where a formal approach is required, the book would be a most useful addition to any teacher's library of mathematics books.

A.I.

Introduction to Higher Mathematics

C. Reid. Routledge & Kegan Paul. 184 pp. 12s. 6d.

This book, by the author of *From Zero to Infinity*, includes a popular account of elementary topics from number theory, complex numbers, non-Euclidean Geometry, topology, calculus and mathematical logic. A better title might have been *Glimpses of Higher Mathematics*, since someone looking for an introduction to a rigorous study of the subject will find little help here. The treatment is cursory and there are no suggestions for further reading. Use of vocabulary is loose; for example, on page 51 we read of an icosahedron being rotated to positions 'so that after each rotation it occupies the same volume as it did before the rotation'. There are interesting historical references which give background to the work, but there are many irritating 'Americanisms' which detract, e.g. 'a pesky question', 'the laws of Arithmetic for cookies'. This book may be of interest to 15-16 year olds, or to non-mathematicians who are interested to know a little of what higher mathematics does.

C.B.

Modern Geometry and Trigonometry (Stage One)

L. Levy. Longmans, Green & Co. Ltd. 94 pp. 6s. 6d.

Teachers frequently enquire for a book which will help them to make a new approach to some branch of the subject; those looking for such to provide 'Stage A' Geometry in lower forms of selective schools or to form the basis of a geometry course in a Secondary Modern School would be well-advised to examine this book.

Mr. Levy sets out, by using lots of practical examples, to lead the pupil to an appreciation of shape, size, area and volume, and by measurement of lines and angles, drawing, working out puzzles and the like, tries to bring an understanding of basic geometrical fact. His approach is imaginative and essentially practical. For example, lengths in geometry are used in this book to provide an introduction to algebra.

One may make criticisms. Does the book try to do too much of the teaching? Rules are introduced into the methods of solving equations without justification or explanation.

But the author has made a commendable effort and we look forward to seeing the other two books of the series. It would probably be helpful to the teacher using these books, if the author concluded his series with a teacher's book containing further suggestions for practical activity in class to supplement the work in the textbooks.

C.B.

Solid Geometry

J. L. Simpson. Harper & Bros., New York, 1960. pp. 97. Price 20/-

This book which covers most of our 'A' level solid geometry and mensuration, was designed for a thirty period course for those specializing in engineering and science.

The first chapter covers most of the theorems on line and plane, giving the usual (Euclidean) proofs. The examples are mostly concerned with intersections and lines through points parallel to planes, etc. As such they form interesting and informative discussion points. The majority of the exercises in the remainder of the book are concerned with mensuration. The usual theorems on prisms and pyramids are given and a weak form of Cavalieri's theorem applied to prisms and pyramids enables the base-height formula to be generalised to cases other than 'right' ones. The chapter on the sphere deals with spherical triangles and their areas, quoting the usual mensurational formula without proof. The book closes with summaries of the formulae, a brief discussion of dihedral angles, formulae and the five regular polyhedra. Euler's Theorem and the nets of regular solids are omitted.

The examples are very simple, the explanation of the text is very lucid and the diagrams are remarkably clear and the whole book is well printed. U.S.A. syllabuses are reflected in the fact that all algebraic requirements are reduced to a minimum. Only few applications are given and those in the examples. There is a feeling of

abstractness about the whole book although a pupil could read it without much tutorial help. It would form a suitable mathematical reference book for 'O' level courses in all types of secondary schools.

C.H.

Mathematical Puzzles from 'Scientific American'

Martin Gardner. Bell 1961. pp. x + 163 price 17s. 6d.

Those who have read Scientific American will need no further information about this book: they will go out and buy it!

The author has assembled a truly enthralling collection of puzzles, paradoxes and mathematical games. I could not put it down once I had begun to read it. Hexaflexagons, probability paradoxes, problems, card tricks, fallacies, noughts and crosses, hex, polynominoes and topological models provide interesting essays, well-written and supplemented by selections from the correspondence which followed their publication. Anyone would have to be a mathematical moron not to work through this book with excitement and interest. Buy it for your pleasure, buy it for your school—many a lesson will be inspired by it and many of your pupils will want to read it.

I detected only one minor blemish, the triangle printed on page 133 should be one square greater in altitude.

C.H.

Introduction to Modern Algebra

John L. Kelley. D. Van Nostrand Co. Inc. pp. x + 338. 21/-

Student Manual

Roy Dubitsch. D. Van Nostrand. pp. 63. 7s. 6d.

This book was written for the 'Continental Classroom' in the U.S.A. as a private study book to be read in conjunction with the book of assignments, the Student Manual by Dubitsch. The manual makes out suitable lists of examples with commentaries which tie together some of the ideas round which the assignment is centred.

It is a very readable introduction to the ideas of algebras, vectors and their applications to euclidean spaces. The standard reached is not very advanced; it is, for the greater part, a reappraisal of much of traditional algebra including the axioms of number systems, elementary treatment of vectors and vector spaces of 1, 2, 3 dimensions, complex numbers, linear algebra and elementary theory of matrices mainly treated numerically.

As one of the first of the elementary texts for the new U.S. school programmes written from the 'modern' mathematical viewpoint, the book stands in an invidious position: it has no tradition to justify the selection of topics, it has no anthology of applications at elementary level ready to hand to draw on and its degree of rigour has not yet been tested in a school situation. It is a very readable book, has the right amount of stage A but does not go far enough into stage B. In fact a second volume

which went further into vector spaces, treating bases, dealt with affine and bilinear transformations, would add to the value of the book enormously. So many topics fail to go on developing just as they are getting meaty; solution sets of inequalities, vectors, inner products and linear algebra are examples of this. But having said this, one must remark on the many valuable features of the book: the many examples of applications, the interesting exercises very well graded, the treatment of hyper-complex numbers (quaternions), the very clear, lucid introduction to axiomatics and elementary set theory, inner products and the development of trigonometry. It seems a very reasonable approach for school boys, makes frequent appeals to intuition in justification of a development and would be suitable for the age range 14 to 16.

Many training colleges might find in this introduction a useful textbook for their new three year courses, one which puts school mathematics into a new perspective and as a basis for the meatier courses with which university degrees are concerned. Those teachers who wish to read about the new ideas will find the text and the manual good companions.

The paper covers of the book account for its cheap price. If one does not mind such bindings in the library, a copy would be well worth a place for the sixth form's benefit.

C.H.

Work This One Out

L. H. Longley-Cook. Benn, 1961. 96 pp. 8s. 6d.

This is a collection of mathematical puzzles which will appeal to all interested in such books—and who isn't?

The 105 problems include one or two old favourites and many new ones, and range from the easy (no pencil and paper required) to the difficult (thank heaven for the answers!)—fortunately classified in four degrees of difficulty.

This is a useful little source book for the odd problem to add spice to the maths lesson, and should be in the mathematics library.

C.B.

Common Entrance Arithmetic and Algebra

D. G. Munir; Methuen & Co. Ltd. pp. 235 9s. 6d. With answers, 12s. 6d.

This is a book of examples, no book work is given and the method of approach is left to the teacher. It aims at providing a text book to cover the five years' work from 8 years to 13 years based on the syllabus laid down by the report of the H.M.C. and I.A.P.S. on *The Curriculum for Preparatory Schools*. This it does by some 5000 examples. The nature of the examples has been dictated by the questions set in the Common Entrance Examination. One or two strike one as somewhat out of touch with realities. Why should anyone be asked to add up compound quantities such as mi. fur. ch. yd. ft. in. or bush. pk. gall. qt. pt. ?

The approach to Algebra is unsatisfactory. It is the approach via arithmetic and quickly reveals itself to be nothing more than the Four Rules method condemned by the Mathematical Association's Report on *The Teaching of Algebra*, and in the 28th example we are asked to find $-3x + 2y$ when x is 2 and y is 1 without any mention of negative numbers having been made. In fact the author shows no awareness that there is any difficulty in dealing with directed numbers and nowhere are there any examples which help with this difficulty.

The section on graphs is interesting but here, of the 27 examples, only in 8 of them is the boy asked to draw a graph himself. The emphasis on interpretation is of course dictated by the Common Entrance papers, but it would be better in a book like this that pupils be guided to do something constructive as well as interpretive.

The printed graphs are taken from Common Entrance papers and this is a pity. They are inaccurately drawn and all seven quadratic graphs look like Gothic windows, or worse, and give an entirely wrong idea of what a parabola really looks like. This is unfortunate as it is the only section of the book that even approaches any idea of functionality.

The number of examples in the form of Arithmetic Problems is adequate but the numbers of routine examples is totally inadequate. For instance, in the section covering products of binomials there are a mere 66 examples.

The most serious criticism is that an Arithmetic approach to Algebra leads to sterility and one cannot help thinking that boys who follow this course will have little conception of the true functional nature of the subject.

R.D.K.

Mathematics and the Physical World

Morris Kline. John Murray, 1960 pp. ix + 482 25s.

The purpose of this book is "to display the role of mathematics in the study of nature". The contents range from arithmetic through algebra, co-ordinate geometry, and calculus to non-Euclidean geometry, gathering in on the way, light theory, Newton's laws of motion, orbital mechanics and oscillatory motion.

It succeeds in giving a very good explanatory account of the interrelationship between science and mathematics. Every teacher and most sixth-formers would do well to read it. In accordance with good teaching practice each topic is first discussed in an elementary "stage A" way followed by a "stage B". In many places one feels that the author strains to explain mathematical ideas which a normal reader would have already grasped. There is a good leavening of philosophy and frequent appeal to historical methods which avoid the mathematics of later generations. For example Huygen's treatment of planetary motion brings it within the grasp of O level students.

Some blemishes are to be detected: on page 2, it seems to me there is an error and on page 246 a constant acceleration is used without any attempt at rationalisation to derive the acceleration of a body moving in a circle.

One may thoroughly recommend this book as a welcome addition to the literature which makes mathematics real to everyman. C.H.

A History of Mathematics for Secondary Schools

H. A. Freebury. Cassell and Co. Ltd., London, 1958. pp. x + 198. price 8s. 6d.

This book has enjoyed a deserved popularity since its publication. It was written to cover the history of mathematics syllabus of G.C.E. and is constructed with the answering of examination questions very much in mind.

Its material is rather naturally drawn from secondary sources and current literature. Like most other books at this level it is a book about the development of mathematical ideas without actually ever doing any mathematics. A little more detail and a little more mathematics would have enhanced its value immensely, not only making it more interesting but also very useful background to O level courses.

As it stands it is a good introduction to many topics which could then be followed up in a good library. C.H.

General Certificate Mathematics Tests

W. A. Gibby. Odhams Press. 80 pages. 4s. With answers, 4s. 9d.

This collection of mathematics questions will appeal to those teachers preparing students for 'O' level G.C.E. on Syllabus B of the various examining bodies who require suitable examples for pre-examination revision. There are over 300 examples in the book covering every topic likely to be found at this level. A particular feature of the book is that the questions are arranged in test papers of 40 minutes duration, making them particularly suitable for use in one lesson or as a homework. The tests were standardised by giving some of them to over 1,300 children, so that they enable the teacher to form a good opinion of G.C.E. performance. C.B.

Graphs for Interpretation

Gordon L. Bell. Harrap, 1960 pp. 91 price 7s. 6d.

This is a book of conventional graphs: pictographs, column graphs, line graphs, jagged line graphs, curved graphs, straight line graphs and time and distance graphs. Each topic has a number of full page illustrations with questions about them on the opposite page and finishes with a number of exercises.

Graphs are treated very much as a topic, opportunities for developing mathematical principles are lost, and there seems to be no thought of the ways in which these graphs are to be used as symbols in the development of mathematics generally. For those schools who follow a traditional syllabus, this book will undoubtedly be useful although were a filmstrip of the illustrations to be available, the value of the book would be enhanced.

What one would have liked to have seen, would have been some illustrations of normal distributions, some illustrations readily applicable to the idea of gradient.

The printing and clarity are excellent. C.H.

ANNOUNCEMENTS

(Also see page 59)

A **London Branch** of the Association has been formed. The next meeting will be in October; details from Mr. D. Fielker. Particulars of Autumn meetings of other Branches are not to hand as we go to press, but most are arranging meetings and details may be obtained from the local Secretaries whose addresses appear on our back cover.

Back Numbers of *Mathematics Teaching* and *Mathematics Teaching Pamphlets* are now on sale from Mr. J. Tyrer, 4 Downs Road, Maidstone, Kent, and all orders should be sent direct to him (Details on page 59).

Number apparatus. The Secretary frequently has enquiries from teachers who are interested in some form of structural material (mostly Cuisenaire and Stern) asking for advice. In many cases the enquirers would like to be shown the use of the materials or to visit a school using them. Any readers who can help occasionally in either or both of these ways are asked to inform the Secretary.

CHANGE OF TREASURER

Please note that the Treasurer of the Association is now **Mr. I. Harris, 122 North Road, Dartford, Kent.**

HAVE YOU CHANGED YOUR ADDRESS?

Members are reminded that they should inform the Secretary immediately should they change their address. The Association cannot be responsible for copies of *Mathematics Teaching* which may go astray through failure to give such notification.

INCREASE IN SUBSCRIPTION

As reported elsewhere in this issue, the Annual General Meeting on April 13th, 1961, approved the raising of the annual subscription to the Association to one pound (£1) from October 1st, 1961.

The Committee has decided that it will introduce the following additional measures:

(1) A special annual rate of ten shillings (10/-) for accredited student members will be created.

(2) "Mathematics Teaching" will be published four times a year.

(3) The Treasurer will pay two shillings and sixpence (2/6) to an authorised Branch of the Association on behalf of each Branch member who is a fully-paid-up member of the main Association. This money, where granted, will be used to reduce the Branch subscription by a corresponding amount.

(4) Differential charges between members and non-members will be established for some of the Association's services.

The first measure will come into effect from October 1st, 1961; the others from January 1st, 1962.

D.H.W.

Our Next Issue

★ Leicestershire Experiment
(Dienes' apparatus)

★ Developments in Mathematics
as seen from Canada

★ Secondary Modern School Mathematics

★ Models

★ Reviews

AND DON'T FORGET!
With our next issue we commence publication FOUR times a year